VRSVFMF2026-01-09

Slučajni vektorji

$$(X,Y)(\omega) = (X(\omega), Y(\omega))$$

X, Y na istem verjetnostnem prostoru.

Diskretni slučajni vektorji: $P((X,Y)=(x,y)) = P(X=x, Y=y)$

(skupna ali navzkrižna porazdelitev)

+ tabela

$$P(X=x) = \sum_y P(X=x, Y=y)$$
$$P(Y=y) = \sum_x P(X=x, Y=y)$$

Večino poseben primer tabele je: X in Y sta neodvisni, t.j.:

$$P(X=x, Y=y) = P(X=x)\cdot P(Y=y).$$

Zgled: dvakrat neodvisno vržemo pošteno kocko.

$X :=$ št. metov, ko pade šestica

$Y :=$ št. metov, ko pade sodo mnogo pik.

$(X,Y) \sim ?$ $\quad X \sim ?$ $\quad Y \sim ?$

$X \sim \mathrm{Bin}(2, \frac{1}{6})$

$Y \sim \mathrm{Bin}(2, \frac{1}{2})$

$X+Y \sim ?$

Če $X' \stackrel{d}{=} X$ in $Y' = Y$, ki sta neodvisni

pomeni $\exists$ porazdelitev P $\exists$: $X' \sim P$ in $X \sim P$

robni porazdelitvi ostaneta, vrednosti celic znova izračunamo glede na

	$Y'=0$	$Y'=1$	$Y'=2$	
$X'=0$	$\frac{25}{144}$	$\frac{50}{144}$	$\frac{25}{144}$	$\frac{25}{36}$
$X'=1$	$\frac{10}{144}$	$\frac{20}{144}$	$\frac{10}{144}$	$\frac{5}{18} = \frac{10}{36}$
$X'=2$	$\frac{1}{144}$	$\frac{2}{144}$	$\frac{1}{144}$	$\frac{1}{36}$
	$\frac{1}{4}$	$\frac{1}{2} = \frac{2}{4}$	$\frac{1}{4}$	1

tedaj

$$X' + Y' \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{25}{144} & \frac{60}{144} & \frac{46}{144} & \frac{12}{144} & \frac{1}{144} \end{pmatrix}$$

N: KRIŽANKA: dopolnite z rdečo tabelo, vedoč, da sta X in Y neodvisni.

	$y=0$	$y=1$	$y=2$	
$X=-1$	0,05	0,1	0,1	0,25
$X=0$	0,1	0,2	0,2	0,5
$X=1$	0,05	0,1	0,1	0,25
	0,2	0,4	0,4	1

X in Y sta neodvisni $\Leftrightarrow$ matrika njunih porazdelitev ima rang 1.

N

$X \sim Poi(\alpha)$: $P(X=k) = \dfrac{\alpha^k e^{-\alpha}}{k!}$ za $k = 0,1,2,\dots$

aproksimacija za Binomsko porazdelitev pri redkih dogodkih.

asistent poda primer: „št. mrtvih vojakov vsled brce konja".

$S \sim Poi(\lambda)$ $T \sim Poi(\mu)$ neodvisni izračunaj $S+T \sim ?$

$n = 0, 1, 2, \dots$

$$\{S+T=n\} = \bigcup_{k=0}^{n} \{S=k \,;\, T=n-k\}$$

disjunktna

$$P(S+T=n) = \sum_{k=0}^{n} P(S=k, T=n-k) = \sum_{k=0}^{n} P(S=k)\cdot P(T=n-k) =$$

$$= \sum_{k=0}^{n} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{\mu^{n-k} e^{-\mu}}{(n-k)!} = e^{-(\lambda+\mu)} \sum_{k=0}^{n} \frac{\lambda^k \mu^{n-k}}{k!(n-k)!} =$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^k \cdot \mu^{n-k} = \frac{e^{-(\lambda+\mu)}}{n!} (\lambda+\mu)^n \sim Poi(\lambda+\mu)$$

Vzemimo $\left.\begin{matrix} S_1, S_2, S_3, \dots \\ T_1, T_2, T_3, \dots \end{matrix}\right\}$ naj bodo neodvisne

in $S_k \sim Bin(m_k, p_k)$

in $T_k \sim Bin(n_k, p_k)$

Vemo $S_k + T_k \sim Bin(m_k + n_k, p_k)$

naj $p_k \to 0$, ; $m_k p_k \to \lambda$; $n_k p_k \to \mu$

tedaj $S_k \to Poi(\lambda)$

$T_k \to Poi(\mu)$

$S_k + T_k \to Poi(\mu+\lambda)$

---

neodvisnost v splošnem:

$\forall A, B$ merljivi borelovi $\subseteq \mathbb{R}$: $P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

zvezno porazdeljeni slučajni vektorji:

$$P((X,Y) \in A) = \iint_A \underbrace{p_{X,Y}(x,y)}_{\text{skupna gostota}}\, dx\,dy$$

$$\iint_{\mathbb{R}^2} p_{X,Y}(x,y)\,dx\,dy = 1$$

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$$

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx$$

zato (X,Y) zvezno por. $\Rightarrow$ X in Y posebej zvezno p.

let $p_{X,Y}(x,y) = \begin{cases} ce^{-x} & ; \ x > 2y > 0 \\ 0 & ; \ \text{sicer} \end{cases}$

$c = ?$ $\quad p_X, p_Y = ?$ $\quad P(X > 3Y) = ?$

$P(3X > Y) = ?$ Sta X in Y neodvisni?

$$\iint_{x>2y>0} ce^{-x}\,dx\,dy = \int_0^\infty dy \int_{2y}^\infty ce^{-x}\,dx = \int_0^\infty \left(-ce^{-x}\right)\Big|_{2y}^{\infty} dy = \int_0^\infty ce^{-2y}\,dy =$$

$$= \left(-\frac{1}{2}ce^{-2y}\right)\Big|_0^\infty = \frac{1}{2}ce^0 = \frac{1}{2}c = 1 \Rightarrow \underline{c = 2}$$

$$\underset{y>0}{p_Y(y)} = \int_{2y}^\infty 2e^{-x}\,dx = \left(-2e^{-x}\right)\Big|_{2y}^{\infty} = 2e^{-2y}$$

$$p_Y(y) = \begin{cases} 2e^{-2y} & ; \ y > 0 \\ 0 & ; \ \text{sicer} \end{cases} \Rightarrow Y \sim \text{Exp}(2) = \text{Gama}(1,2)$$

$$\underset{x>0}{p_X(x)} = \int_0^{1/2x} 2e^{-x}\,dy = 2e^{-x} \cdot \frac{1}{2}x$$

$$p_X(x) = \begin{cases} xe^{-x} & ; \ x > 0 \\ 0 & ; \ \text{sicer} \end{cases} \Rightarrow X \sim \text{Gama}(2,1)$$