OMPFMF 2025-05-16

let $f: K \xrightarrow{konv} \mathbb{R}$. $f$ konv, če je $\forall x,y \in K$, $\forall \lambda \in [0,1]$,

$$f((1-\lambda)x+\lambda y) \leq (1-\lambda)f(x)+\lambda f(y)$$

primer: $f(x)=a^T x + b$ je afina fja (linearna + konst)

$$f(x) = a_1x_1+\dots+a_nx_n+b$$

je konveksna:

leva stran $f((1-\lambda)x+\lambda y) = a^T((1-\lambda)x+\lambda y)+b =$

velja enakost $=(1-\lambda)a^Tx+\lambda a^Ty+b$

desna stran $(1-\lambda)f(x)+\lambda f(y) = (1-\lambda)(a^Tx+b)+\lambda(a^Ty+b) =$

$=(1-\lambda)a^Tx+\lambda a^Ty+b$

$\Rightarrow$ afina fja je hkrati konveksna in konkavna hkrati

Trditev: $f: K \xrightarrow{konv} \mathbb{R}$ konveksna. let $b \in \mathbb{R}$ poljuben

let $A = \{x \in K : f(x) \leq b\}$. potem je $A$ konveksna.

Dokaz: $x,y \in A$, $\lambda \in [0,1]$. $\underline{(1-\lambda)x+\lambda y \in A}$

$$f((1-\lambda)x+\lambda y) \underbrace{\leq}_{\text{konveksnost}} \underbrace{(1-\lambda)}_{\geq 0}\underbrace{f(x)}_{\leq b}+\underbrace{\lambda}_{\geq 0} \underbrace{f(y)}_{\leq b} \leq (1-\lambda)b+\lambda b = b$$

Trditev: (1) $f,g: K \xrightarrow{konv} \mathbb{R}$ konv. $\Rightarrow f+g$ konv

$c \geq 0 \Rightarrow c \cdot f$ konv.

(2) $f: g(K) \to \mathbb{R}$, $g: K^{konv} \to \mathbb{R}$ afina

$\Rightarrow g(K)$ konveksna.

$f$ konv. potem je $f \circ g$ konv.

(3) $f: C(g(K)) \to \mathbb{R}$ konveksna in naraščajoča,

$g: K^{konv.} \to \mathbb{R}$ konveksna

Potem je $f \circ g$ konveksna.

<u>Dokaz:</u> (1) $x, y \in K$, $\lambda = [0,1]$

$$(f+g)((1-\lambda)x + \lambda y) = f((1-\lambda)x + \lambda y) + g((1-\lambda)x + \lambda y)$$

$$\leq (1-\lambda)f(x) + \lambda f(y) + (1-\lambda)g(x) + \lambda g(y) =$$

$$= (1-\lambda)(f+g)(x) + \lambda(f+g)(y)$$

in

$$c \cdot f((1-\lambda)x + \lambda y) \leq c((1-\lambda)f(x) + \lambda f(y)) =$$

$$= (1-\lambda)cf(x) + \lambda c f(y)$$

(2) $g(x) = a^T x + b$

$$(1-\lambda)g(x) + \lambda g(y) = g((1-\lambda)x + \lambda y) \Rightarrow g(K) \text{ konveks.}$$

$$f \circ g((1-\lambda)x + \lambda y) = f((1-\lambda)g(x) + \lambda g(y)) \leq$$

$$\leq (1-\lambda)f(g(x)) + \lambda f(g(y))$$

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(3) $g((1-\lambda)x+\lambda y) \le (1-\lambda)g(x)+\lambda g(y)$ /f

$$f\circ g((1-\lambda)x+\lambda y) \le f((1-\lambda)g(x)+\lambda g(y))$$

$$\le (1-\lambda)f(g(x))+\lambda f(g(y))$$

ne-primeri: $cf$ ni nujno konveksna

$f(x)=x^2$, $c=-1$

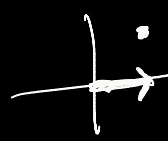
$f\cdot g$ ni nujno konveksna:

$f(x)=x^2$, $g(x)=-1$

$f\circ g$ ni nujno konveksna

$g(x)=x^2$ $f(x)=-x$

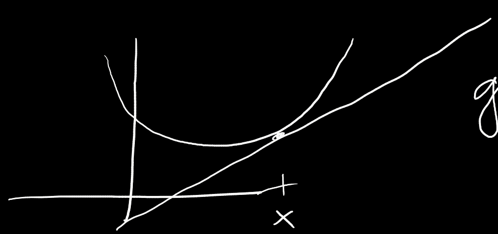
$f(t)$ ni nujno konveksna

 je konv., a slika je $\{0,1\}$

[Kriterija 1. in 2. odvoda]

Za $n=1$:

1. odvod:




graf je nad tangento

$$f(y)\ge f(x)+f'(x)(y-x)$$

$$f''(x)\ge 0 \quad \forall x\in K$$

Je pa zajebano, da rabimo odvedljivost za ta pogoj

**Def.:** gradient f-je $f: \Omega^{odp} \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$

...

$$\nabla f = \operatorname{grad} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

**Izrek:** (kriterij 1. odvoda)

$f: K \to \mathbb{R}$, $K \subseteq \mathbb{R}^n$ odprta, konveksna.

$f$ odvedljiva (tj. $\exists \frac{\partial f}{\partial x_i}: K \to \mathbb{R}$)

Dokaži, da je $f$ konveksna $\Leftrightarrow f(y) \geq f(x) + (\nabla f(x))^T \cdot (y-x)$ $\quad \forall x, y \in K$

**Dokaz:**

$(\Leftarrow)$: $x, y \in K$, $\lambda \in [0,1]$. $\quad z = (1-\lambda)x + \lambda y \in K$

$$f(x) \geq f(z) + (\nabla f(z))^T (x-z)$$
$$f(y) \geq f(z) + (\nabla f(z))^T (y-z)$$

konv. komb.

$$(1-\lambda) f(x) + \lambda f(y) \geq (1-\lambda)\left(f(z) + (\nabla f(z))^T (x-z)\right) + \lambda\left(f(z) + (\nabla f(z))^T (y-z)\right) =$$

$$= \underbrace{(1-\lambda) f(z) + \lambda f(z)}_{= f(z)} + (\nabla f(z))^T \underbrace{\left((1-\lambda)(x-z) + \lambda(y-z)\right)}_{(1-\lambda)x + \lambda y - (1-\lambda)z - \lambda z = 0}$$

$*$

$(\Rightarrow)$ $\quad g: (-\delta, 1] \to K$

$$\lambda \mapsto (1-\lambda)x + \lambda y$$

ker $K$ odprta, $\exists \varepsilon > 0 \ \exists: \|z - x\| < \varepsilon \Rightarrow z \in K$

$$\|(1-\lambda)x + \lambda y - x\| = \|\lambda(y-x)\| = |\lambda| \|y-x\| < \varepsilon \Leftrightarrow$$
$$|\lambda| < \frac{\varepsilon}{\|y-x\|} = \delta$$

$y - x \neq 0$, sicer

$*$ očitno velja.

sedaj vzemimo $f\circ g : [-\delta, 1] \to \mathbb{R}$ in

$(f\circ g)'(0) = ?$

$$f(g(\lambda)) = f((1-\lambda)x+\lambda y) = f((1-\lambda)x_1+\lambda y_1, \dots, (1-\lambda)x_n+\lambda y_n) \quad \Big/ \frac{d}{d\lambda}$$

$$\frac{\partial}{\partial \lambda} f(g(\lambda)) = \frac{\partial f}{\partial x_1}((1-\lambda)x+\lambda y)\cdot(y_1-x_1) + \dots + \frac{\partial f}{\partial x_n}((1-\lambda)x+\lambda y)\cdot(y_n-x_n) =$$

$$= \nabla f((1-\lambda)x+\lambda y)^T\cdot(y-x) \qquad \square$$

Izrek (kriterij odvoda za $n=1$):

Let $f:(a,b)\to\mathbb{R}$ dvakrat odvedljiva

tedaj je $f$ konveksna $\iff f''(x)\geq 0 \ \forall x\in(a,b)$

Dokaz:

$(\Rightarrow)$ $f((\underset{\substack{\downarrow\\ x+h}}{1-\lambda})x+\overset{\substack{1/2\\ \uparrow}}{\lambda} \underset{\substack{\downarrow\\ x-h}}{y}) \leq (1-\lambda)f(x)+\lambda f(y)$

$h$ mora biti dovolj majhen, da sta $x\pm h\in(a,b)$

$$f\left(\tfrac{1}{2}(x+h)+\tfrac{1}{2}(x-h)\right) = f(x) \leq \tfrac{1}{2}f(x+h)+\tfrac{1}{2}f(x-h)$$

velja $0 \leq \lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} \overset{L.H.}{=}$

$$\overset{L.H.}{=} \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \overset{L.H.}{=} \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2} \overset{\text{zveznost}}{=} f''(x)$$

$(\Leftarrow)$ $x, y \in (a,b)$, $x < y$, $\lambda \in (0,1)$

$z := (1-\lambda)x + \lambda y \in (x,y)$

po lagrangeu $\exists \xi_1 \in (x,z) \ni: f'(\xi_1) = \dfrac{f(z)-f(x)}{z-x}$

$\exists \xi_2 \in (z,y) \ni: f'(\xi_2) = \dfrac{f(y)-f(z)}{y-z}$

$\exists \xi_3 \in (\xi_1, \xi_2) \ni: f''(\xi_3) = \dfrac{f'(\xi_2)-f'(\xi_1)}{\xi_2 - \xi_1} \geq 0$

oziroma $f'(\xi_1) \leq f'(\xi_2)$.

ali $f'$ je naraščajoča, ker ima nenegativen odvod, od koder sledi

$z - x = \lambda(y-x)$

$y - z = y - (1-\lambda)x - \lambda y =$

$= (1-\lambda)(y-x)$

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}$$

$$\frac{f(z)-f(x)}{\lambda \cancel{(y-x)}} \leq \frac{f(y)-f(z)}{(1-\lambda)\cancel{(y-x)}} \qquad / \cdot (\lambda)(1-\lambda)$$

$$(1-\lambda)(f(z)-f(x)) \leq \lambda(f(y)-f(z))$$

$$f(z) \leq (1-\lambda)f(y) + \lambda f(y)$$

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Def.: let

$A \in \mathbb{C}^{n \times n}$ $\quad Ax = \lambda x,\ x \neq 0$. tedaj je

$x$ lastni vektor in $\lambda$ lastna vrednost.

lastne vrednosti so ničle karakterističnega polinoma $\quad p_A(\lambda) = \det(A - \lambda I)$

Def.:

A je diagonalizabilna, če $\exists n$ linearno neodvisnih lastnih vektorjev.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ni diagonalizabilna:

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 \quad ; \quad \lambda = 0 \text{ je edina lastna vrednost.}$$

lastni vektorji:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y = 0 \quad x = 0$$

imamo le en lastni vektor $\Rightarrow$ matrika ni diagonalizabilna.

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ je realna matrika s kompleksnimi lastnimi vrednostmi $\Rightarrow$ ni diagonalizabilna

<u>Def.:</u> $A \in \mathbb{R}^{n \times n}$ je simetrična $\Leftrightarrow A^T = A$.

<u>Izrek:</u> potem velja (1) lavr so realne

(2) lave, ki pripadajo različnim lavr so ortogonalni

(3) A je diagonalizabilna, celo v ortonormirani bazi.

<u>Dokaz:</u> (1) $Ax = \lambda x$, $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^n$

$$\overline{x}^T A x = \overline{x}^T(\lambda x) = \lambda \overline{x}^T x = \lambda \|x\|^2$$

$$\|$$

$$x^T \overline{A}^T x = \overline{(Ax)}^T x = \overline{(\lambda x)}^T x = \overline{\lambda} \|x\|^2$$

ker $x \neq 0 \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$

(2) $Ax=\lambda x$, $\lambda\in\mathbb{R}$, $x\in\mathbb{R}^n$ po predpostavki $\mu\neq\lambda$

$Ay=\mu y$ $\mu\in\mathbb{R}$, $y\in\mathbb{R}^n$

$$y^TAx = y^T(\lambda x) = \lambda y^Tx$$
$$\parallel$$
$$y^TA^Tx = (Ay)^Tx = (\mu y)^T = \mu y^Tx$$

$$\Rightarrow \underbrace{(\lambda-\mu)}_{\neq 0} y^Tx = 0 \Rightarrow y^Tx = 0$$

(3) A gotovo ima lastno vrednost $\in\mathbb{R}$ in lahko $x$. Vzemimo vse vektorje, ki so pravokotni na $x$.

$V := \{y\in\mathbb{R}^n ; x^Ty = 0\}$ iz tega sledi,

da je $y\in V \Rightarrow Ay\in V$ (t.j. $V$ je invarianten za $A$)

1° predpostavki $x^Ty=0 \Rightarrow$

$$\Rightarrow x^TAy = x^TA^Ty = (Ax)^Ty =$$
$$= (\lambda x)^Ty = \lambda x^Ty = 0$$

zožimo na

$A|V$ je spet simetrična in nadaljujemo.

ZOB
$$A = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & \end{array}\right]$$

Diagonalizabilnost ZOB pomeni da $\exists$ obrnljiva P, diagonalna D

$\in \mathbb{C}^{n\times n}$ $\exists$: $A = PDP^{-1}$

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za simetrično $A = A^T$ velja pa

$$A = QDQ^T \quad \text{za} \quad \underbrace{Q^T = Q^{-1}}_{\text{ortogonalna matrika}}$$