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QR-razcep

$A \in \mathbb{R}^{m\times n}$, $m \geq n$

$A = QR$, $Q \in \mathbb{R}^{m\times n}$ z ortonormiranimi stolpci ~ $Q^T Q = I$

$R \in \mathbb{R}^{n\times n}$ zgornje3kotna s pozitivnimi diagonalci

Gram-Schmidtova diagonalizacija ~ Modificiran Gram-Schmidtov postopek

$k = 1..n$:

$\quad q_k = a_k$

$\quad j = 1..k-1$:

$\quad\quad r_{jk} = q_j^T q_k$

$\quad\quad q_k = q_k - r_{jk} q_j$

$\quad r_{kk} = \|q_k\|_2$

$\quad q_k = \frac{1}{r_{kk}} q_k$

Pri reševanju predoločenega sistema $Ax = b$ z MGS naredimo QR-razcep na razširjeni matriki

$$[A \,\vdots\, b] = [Q \,\vdots\, q_{n+1}] \begin{bmatrix} R & z \\ 0 & \rho \end{bmatrix}$$ in

rešujemo $Rx = z$, $\min_x \|Ax-b\|_2 = \rho$

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Preko modificiranega GS določite rešitev sistema $Ax=b$ po metodi najmanjših kvadratov, če je

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 4 \\ 6 \\ -1 \\ 2 \end{bmatrix}$$

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Razširjeni QR razcep

$A \in \mathbb{R}^{m\times n}$, $m \geq n$

$A = \tilde{Q}\tilde{R}$: $\tilde{Q} \in \mathbb{R}^{m\times m}$ ortogonalna

$\tilde{R} \in \mathbb{R}^{m\times n}$ zgornja trapezna $\begin{bmatrix} * \\ 0 \end{bmatrix}$

$$A = \tilde{Q}\tilde{R} = [Q, Q_1] \begin{bmatrix} R \\ 0 \end{bmatrix}$$

reševanje sistema: iščemo x, $\min_x \|Ax-b\|_2$.

$$\|\tilde{Q}\tilde{R}x - b\|_2 = \|\underbrace{\tilde{Q}^T}_{\text{množenje z ortog. matriko ne spremeni } \|\cdot\|_2}(\tilde{Q}\tilde{R}x - b)\|_2 = \|\cancel{\tilde{Q}^T\tilde{Q}}\tilde{R}x - \tilde{Q}^T b\|_2 =$$

$$= \|\tilde{R}x - \tilde{Q}^T b\|_2 =$$

$$= \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} x - \begin{bmatrix} Q^T \\ Q_1^T \end{bmatrix} b \right\|_2$$

To bo minimalno, ko $Rx = Q^T b$ in tedaj minimum je $\|Q_1^T b\|$

Givensove rotacije

Vhod: A, b

$k = 1..n$:

$j = k+1, \dots, m$:

$r = \sqrt{a_{kk}^2 + a_{jk}^2}$

$c = \frac{a_{kk}}{r}$

$s = \frac{a_{jk}}{r}$

$R_{kj}^T := eye(A.shape)$

$R_{kj}^T[k,k] := c$

$R_{kj}^T[k,j] := s$

$R_{kj}^T[j,k] := -s$

$R_{kj}^T[j,j] := c$

$A := R_{kj}^T \cdot A$

$b := R_{kj}^T \cdot b$

izhod: $A \; (= \tilde{R})$

$b \; (= Q^T b)$

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let $A = \begin{bmatrix} 3 \\ -4 \\ 12 \\ -84 \end{bmatrix}$ $b = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 0 \end{bmatrix}$ z uporabo givensovih rotacij rešite ...

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