SVD za normalne matrike: $L^*L = LL^*$

$A = U \Sigma V^*$

normalna $\Leftrightarrow$ lastni vektorji so ortogonalni in matriko se da diagonalizirati

$A = PDP^{-1} = PDP^*$

$A v_i = \lambda_i v_i \Rightarrow A^* A v_i = \lambda_i A^* v_i$

$A^* v_i = \overline{\lambda_i} v_i \quad A A^* v_i = \overline{\lambda} A v_i$

$$A^* A v_i = \overline{\lambda} \lambda v_i$$

lastne vrednosti A^*A so $|\lambda|^2$,

torej so singularne vrednosti normalne matrike kar norme lastnih vrednosti

Uredimo jih po velikosti:

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_r| \geq |\lambda_{r+1}| = \dots = |\lambda_n| = 0$$

r → rang A

$$V = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \qquad u_i = \frac{1}{\sigma_i} \cdot A v_i = \frac{\lambda_i}{|\lambda_i|} v_i$$

$\forall i \in \{1 .. r\}$:

dopolnitev $u_{r+1}, \dots u_n$ so kar v_i, saj je V ONB in $u_1, \dots, u_r$ so le skalirani v_i:

$$U = \begin{bmatrix} \frac{\lambda_1}{|\lambda_1|} v_1 & \dots & \frac{\lambda_r}{|\lambda_r|} v_r & v_{r+1} & \dots & v_n \end{bmatrix}$$

N

Frobeniusova norma in skalarni produkt:

$$\langle B, C \rangle_F \quad \text{in} \quad \|B\|_F$$

$$\|B\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |b_{ij}|^2} \quad \text{in} \quad \langle B, C \rangle_F = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \overline{c_{ij}} =$$

$$= \operatorname{sled}(BC^*)$$

$$Q_{1,2} Q_{1,2}^* = I$$

$$\Updownarrow$$

a.) Pokaži, da Q_1, Q_2 unitarni $\Longrightarrow$ $\|Q_1^* A Q_2\|^2 = \|A\|^2$

$$\langle Q_1^* A Q_2, Q_1^* A Q_2 \rangle = s\left(Q_1^* A Q_2 (Q_1^* A Q_2)^*\right) =$$

$$= s\left(Q_1^* A \cancel{Q_2 Q_2^*} A^* Q_1\right) = s\left(Q_1^* A A^* Q_1\right) = s\left(Q_1^* Q_1 A A^*\right) =$$

$$= s(AA^*) = \langle A, A \rangle = \|A\|^2 \quad \square$$

davno tega smo dokazali
$\operatorname{sled}(xy) = \operatorname{sled}(yx)$,
če so elementi iz polja.
za elemente iz obsega,
kjer ni komutativnosti,
to ne velja.

<u>Izrek</u>: (Eckart & Young, 1936):
Naj bo $A \in M_{m \times n}(\mathbb{C})$ in
rang $(A) = r$, $k < r$.

$A = U \Sigma V^*$ (SVD)

Def: $\Sigma_k :=$ matrika Σ, kjer $\sigma_{k+1}, \dots, \sigma_r$ zamenjamo z 0

$A_k := U \Sigma_k V^*$ se med vsemi matrikami ranga $\leq k$ najbolj <u>prilega</u> A.

$\hookrightarrow$ najbližje A glede na normo.

Uporaba: slika velikosti $m \times n$ $\to$ $m \cdot n$ podatkov.

let $A \in M^{m \times n}(\mathbb{R})$.

$$A = U \Sigma V^* \rightsquigarrow A_k = U \Sigma_k V^* = \sigma_1 u_1 v_1 + \dots + \sigma_k u_k v_k$$

Primer: $m = n = 200$ $\quad k = 50$

$$\frac{k(m+n+1)}{mn} = \frac{50401}{4000} \approx 0{,}5$$

N

Poišči ortogonalno matriko, ki je najbližje dani realni $n \times n$ matriki A!

naj bo to Q. $\quad \|A - Q\|_F = \|U \Sigma V^T - Q\|_F =$

$$= \|U^T U \Sigma V^T V - \underbrace{U^T}_{\text{ortog.}} Q \underbrace{V}_{\text{ortog.}}\|_F = \|\Sigma - \underbrace{U^T Q V}_{\substack{\text{ortog, ker je} \\ \text{produkt ortog.} \\ \| \\ \tilde{Q}}}\|_F = \|\Sigma - \tilde{Q}\|_F$$

$$\|\Sigma - \tilde{Q}\|_F^2 = \langle \Sigma - \tilde{Q}, \Sigma - \tilde{Q} \rangle =$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \begin{cases} (\sigma_k + \tilde{q}_{kk})^2 & ; k = j \\ 0 - \tilde{q}_{kj}^2 & ; k \neq j \end{cases} =$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \begin{cases} \sigma_k^2 + 2\sigma_k \tilde{q}_{kk} + \tilde{q}_{kk}^2 & ; k = j \\ \tilde{q}_{kj}^2 + & ; k \neq j \end{cases} =$$

$$= \sum_{k=1}^{n} \Big(\underbrace{\sigma_k^2 + 2\sigma_k \tilde{q}_{kk}}_{\text{sled}(\Sigma^2)} + \underbrace{\sum_{j=1}^{n} \tilde{q}_{kj}^2}_{\text{sled}(Q Q^T) = n} \Big) =$$

$$Q Q^T = I$$

$$= \text{sled}(\Sigma^2) + n - 2 \sum_{k=1}^{n} \sigma_k \tilde{q}_{kk} \quad \text{čim večje}$$

$\swarrow$

ker je $\hat{Q}$ z ON stolpci, je lahko $\tilde{q}_{tt}$ največ 1:

$\tilde{\hat{Q}} = I$, sedaj lahko izrazimo

Q:

$$U \cdot \backslash \quad \hat{Q} = U^T Q V \quad / \cdot V^T$$

$$I = U^T Q V$$

$$\underline{UV = Q} \quad \checkmark$$

N

Imejmo preslikavo $L : V \to U$

velja $\quad L^*L$ je sebiadjungirana:

$$L^*L = (L^*L) \quad \checkmark \quad \Rightarrow L^*L \text{ normalna}$$

$\exists$ ONB za V iz lastnih vektorjev za L^*L

$\ker(L^*L) =$ lastni podprostor za lastv. 0

ONB za $\ker(L^*L)$ obstaja: $\{v_{r+1}, \dots, v_n\}$

$= \ker(L)$

$\operatorname{Im}(L^*L)$ je vsota vseh ostalih lapodpr.

ONB za $\operatorname{Im} L^*L$ $\exists$: $\{v_1, \dots, v_r\}$

$\stackrel{LAIP}{=} \ker(L)^\perp \stackrel{LAIP}{=} \operatorname{Im}(L^*)$

$$\forall i \in \{1..r\}: \quad u_i = \frac{1}{\sqrt{\lambda_i}} L v_i \quad + \text{ dopolnimo do ONB za prostor } U.$$

$\Rightarrow$ $u_1, \dots, u_r$ so ONB za $\operatorname{Im} L = \operatorname{Im} LL^*$

$u_{r+1}, \dots, u_n$ so ONB za $(\operatorname{Im} L)^{\perp} \overset{(A)}{=} \operatorname{Ker} L^*$

torej če preslikavi L priredimo matriko glede na ONB, dobimo pravokotno diagonalno matriko s singularnimi vrednostmi po diagonali.

$\Rightarrow$ singularni razcep ni enolićen.

N

POLARNA DEKOMPOZICIJA MATRIKE

1.) let A kvadratna

$$A = U \cdot P \Rightarrow U \text{ unitarna}, \quad P \geq 0$$

$\in \mathbb{C}^{n \times n}$

$$A = Q_1 \Sigma Q_2^* = \underbrace{Q_1 Q_2^*}_{U} \overset{I}{} \underbrace{Q_2 \Sigma Q_2^*}_{P}$$

če produkt unitarnih $\Rightarrow$ unitarna $\checkmark$

$P \geq 0$

$$P = Q_2 \Sigma Q_2^*$$

$$P^* = Q_2 \Sigma^* Q_2^* = P$$

diag

torej $P \sim \Sigma \Rightarrow$ lastv(P) so ≥ 0

$\hookrightarrow P \geq 0$

2. A $m \times n$ matrika: BSŠ $m \geq n$

$$A = Q_1 \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q_2^*$$

bločno

$$Q_1 = [\underbrace{Q_1^1}_{n} \quad \underbrace{Q_1^2}_{m-n}]$$

$A = U \cdot P$

pričakovano, da $\underbrace{U = Q_1^1 V^*}$

ni unitarna, toda je v e.o smer

$P = V \Sigma V^*$

$U^* U = I$

$U U^* \neq I$