N

$A, T: V \to V$: V evklidski prostor, A pozitivno definitna $\tilde{A} > 0$

Dokaži: $(A^{-1} T^* A) T = T (A^{-1} T^* A) \Rightarrow T$ je diagonalizabilen

namig: obstaja $\underbrace{\sqrt{A}}$. → to je funkcija

$A > 0 \Rightarrow \forall i: \lambda_i > 0$

definicija pozitivne definitnosti:
$A > 0 \iff A = A^*$ in $\forall x \in V \setminus \{0\}: \langle Ax, x \rangle > 0$

$$A = UDU^* = U \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \cdot U^* \Rightarrow \sqrt{A} = U \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U^*$$

in velja $\sqrt{A} > 0$.

$\sqrt{A}^{-1}$ obstaja, ker so vse lastne vrednosti pozitivne $\Rightarrow \det \sqrt{A} > 0$

pišimo $A = \sqrt{A} \cdot \sqrt{A}$:

$$\left(\sqrt{A}^{-1} \sqrt{A}^{-1} T^* (\sqrt{A} \cdot \sqrt{A})\right) T = T \left(\sqrt{A}^{-1} \sqrt{A}^{-1} T^* \sqrt{A} \sqrt{A}\right)$$

$\big| \cdot \sqrt{A}^{-1}$ (D) $\quad \big| \cdot \sqrt{A}$ (L)

$$\underbrace{\sqrt{A}^{-1} T^* \sqrt{A}}\, \underbrace{\sqrt{A}\, T \sqrt{A}^{-1}} = \underbrace{\sqrt{A}^{-1} T \sqrt{A}}\, \underbrace{\sqrt{A}^{-1} T^* \sqrt{A}}$$

$$\overset{\parallel}{(\sqrt{A}\, T \sqrt{A}^{-1})^*} = (\sqrt{A}^{-1})^* T^* \sqrt{A}^* = \sqrt{A}^{*\,-1} T^* \sqrt{A}^* = \sqrt{A}^{-1} T^* \sqrt{A}$$

$$\left(\sqrt{A}\,T\,\sqrt{A}^{-1}\right)^* \sqrt{A}\,T\,\sqrt{A}^{-1} = \sqrt{A}^{-1}\,T\,\sqrt{A}^{-1}\left(\sqrt{A}\,T\,\sqrt{A}^{-1}\right)^*$$

torej $\sqrt{A}\,T\,\sqrt{A}^{-1}$ je normalna

$\Rightarrow$ se jo da diagonalizirati:

$$\sqrt{A}^{-1} \backslash \quad \sqrt{A}\,T\,\sqrt{A}^{-1} = \tilde{U}\,\tilde{D}\,\tilde{U}^{-1} \quad /\cdot\sqrt{A}$$

$$T = \underbrace{\left(\sqrt{A}^{-1}\tilde{U}\right)}_{P}\ \underbrace{\tilde{D}}_{\text{diag}}\ \underbrace{\left(\tilde{U}^{-1}\sqrt{A}\right)}_{P^{-1}}$$

N

kompletsna matrika je **hermitska** $\Leftrightarrow$

$\forall v: \langle Av, v\rangle \in \mathbb{R}$

$\rightarrow$ $A = A^*$

$\Leftrightarrow$

$B = A - A^* = 0$

$$\langle Bv, v\rangle = \langle (A-A^*)v, v\rangle =$$

$$= \langle Av, v\rangle - \langle A^*v, v\rangle = \langle Av, v\rangle - \langle v, Av\rangle =$$

$$= \langle Av, v\rangle - \overline{\langle Av, v\rangle}$$

Torej $\langle Av, v\rangle \in \mathbb{R} \Leftrightarrow \langle Bv, v\rangle = 0$

sedaj šele zares začnemo dokazovati:

$(\Rightarrow)$ A hermitska $\Rightarrow B = 0 \Rightarrow \langle Bv, v\rangle = 0 \Rightarrow \langle Av, v\rangle \in \mathbb{R}$

$(\Leftarrow)$ $\langle Av, v\rangle \in \mathbb{R} \Leftrightarrow \langle Bv, v\rangle = 0$ velja za vsak v, torej velja za poljuben $v = u + u'$:

predpostavimo

$$0 = \langle B(u+u'), u+u'\rangle = \langle Bu + Bu', u+u'\rangle =$$

$$= \underbrace{\cancel{\langle Bu, u\rangle}}_{0} + \langle Bu', u\rangle + \langle Bu, u'\rangle + \underbrace{\cancel{\langle Bu', u'\rangle}}_{0} =$$

$$= \langle Bu, u'\rangle + \langle Bu, u'\rangle = 0$$

vstavimo $i \cdot u$ namesto u:

$$0 = \langle Biu, u'\rangle + \langle Bu', iu\rangle =$$

$$= \cancel{i}\langle Bu, u'\rangle - \cancel{i}\langle Bu', u\rangle = 0$$

$$0 = \langle Bu, u'\rangle - \langle Bu', u\rangle$$

$\forall u, u'$: $\langle Bu, u'\rangle = \langle Bu', u\rangle = 0$

vstavimo $u' = Bu$:

$$\langle Bu, Bu\rangle = 0 \Rightarrow Bu = 0 \Rightarrow B = 0$$

$\Rightarrow$ A hermitska.

Nr

Klasificiraj vse 2×2 ortogonalne matrike

realne

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$AA^T=I \qquad \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}=$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\exists \varphi$ $\ni$: $a=\cos\varphi$ $c=\sin\varphi$

$\exists \alpha$ $\ni$: $b=\cos\alpha$ $d=\sin\alpha$

$ab+cd=0$: $\cos\varphi\cos\alpha+\sin\varphi\sin\alpha=0$

$$\cos(\alpha-\varphi)=0$$

$$\alpha-\varphi=\frac{\pi}{2}+k\pi$$

$$\alpha=\frac{\pi}{2}+\varphi+k\pi$$

torej

$$=A=\begin{bmatrix} \cos\varphi & \cos(\frac{\pi}{2}+\varphi+k\pi) \\ \sin\varphi & \sin(\frac{\pi}{2}+\varphi+k\pi) \end{bmatrix}=\begin{bmatrix} \cos\varphi & -(-1)^k\cdot\sin\varphi \\ \sin\varphi & (-1)^k\cos\varphi \end{bmatrix}$$

primeri: t sod:

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

rotacije

t lih:

$$A = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

rotacije z zrcaljenjem

N

Singularni razcepi: Poišči SVD!

a) $A = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$.

$$\underset{m \times n}{A} = \underset{\substack{m\times m \\ \text{ortog}}}{U} \, \overset{m \times n}{\Sigma} \, \underset{\substack{n \times n \\ \text{ortog}}}{V^*}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & 0 \\ & \ddots & & 0 \\ 0 & & \sigma_n & 0 \end{bmatrix}$$

singularne vrednosti.

SV so koreni lastnih vrednosti A^*A.

$$(A^*A)\, v_i = \sigma_i^2 \, v_i$$

$$\forall i \in \{1..\text{rang} A\}: \quad U_i = \frac{1}{\sigma_i} \cdot A v_i$$

TRIK: let $\lambda \neq 0$ lavr za AA^*:

$$AA^* v = \lambda v \quad / A^* \cdot$$

$$A^*AA^* v = A^* \lambda v$$

$$A^*A\underbrace{A^*v}_{u} = \lambda \underbrace{A^*v}_{u}$$

$$A^*Au = \lambda u$$

torej je λ lavr tudi za A^*A.

AA^* in A^*A imata iste neničelne lavr ker vsa ta če polgubo ničel.

Izračunajmo torej lastne vrednosti za AA^*:

$$\begin{bmatrix} 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \end{bmatrix}$$

lastna vrednost je $25 = \sigma_1^2$

torej $\Sigma = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$

$\sigma_1 = 5$

$$U = \begin{bmatrix} 1 \end{bmatrix}$$

ortogonalna in 1×1.

Sedaj še A^*A: $\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix}$

izračunajmo lastne vektorje za $\lambda = 25$:

$\lambda = 25$:

$$(A^*A - 25I) = \begin{bmatrix} -16 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & -9 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y = 0$

$4x = 3z$

$z = \frac{4}{3}x$

še $\lambda = 0$:

$$(A^*A - 0I) = \begin{bmatrix} 9 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$3x = -4z$

$z = \frac{-3}{4}x$

$V_1 = (3, 0, 4)$

$\|V_1\| = \sqrt{9 + 16} = 5$

$\tilde{V}_1 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$ normiran.

$V_2 = (0, 1, 0)$

$V_3 = (4, 0, -3)$

$\|V_3\| = 5$

$\tilde{V}_3 = \left(\frac{4}{5}, 0, \frac{-3}{5}\right)$ normiran

Te vektorje vstavimo v stolpce V:

$$V = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & -3/5 \end{bmatrix}$$

b.) $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$$AA^* = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$$\begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = (17-\lambda)^2 - 64 = \dots = (x-25)(x-9)$$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$\downarrow$
iste velikosti kot A

$$A^*A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$(A^*A - 25I) = \dots \sim \dots \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$z = 0$

$x = y$

$$V_1 = \frac{\sqrt{2}}{2}(1, 1, 0)$$

$$(A^*A - 9I) = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = -4y$

$x = -y$

$$\tilde{V_2} = (1, -1, 4)$$

$$V_2 = 3\sqrt{2}\,(1, -1, 4)$$

Ker smo dobili 2 lastna vektorja, imamo še eno lastno vrednost 0 v 3×3 najti.

$$(A^*A - 0I) = \begin{bmatrix} 13 & 12 & 2 \\ -1 & 1 & -4 \\ 2 & -2 & 8 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \qquad y = 2z = -x$$

$$\tilde{v}_3 = (-2, 2, 1)$$

$$V = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/6 & -2/3 \\ \sqrt{2}/2 & -\sqrt{2}/6 & 2/3 \\ 0 & \frac{4\sqrt{2}}{6} & 1/3 \end{bmatrix} \qquad \|v_3\| = 3 \qquad v_3 = \frac{1}{3}(-2, 2, 1)$$

$\operatorname{rang} A = 2$, potrebujemo 2 u vektorja z 2×2 matriko U

$$\forall i \in \{1..\operatorname{rang} A\}: \quad \boxed{u_i = \frac{1}{\sigma_i} A v_i}$$

OBRAZEC ZA u_i

$$u_1 = \frac{1}{5} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{\sqrt{2}}{10} \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{1}{3} \cdot \frac{\sqrt{2}}{6} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{\sqrt{2}}{18} \begin{bmatrix} 9 \\ -9 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\hookrightarrow$ morajo biti normirani

N

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A^*A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} A =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 4$, $\lambda_4 = 9$

$\sigma_4 = 0$, $\sigma_3 = 1$, $\sigma_2 = 2$, $\sigma_1 = 3$

$$\Sigma = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A^*A - 9I) = \begin{bmatrix} 9 & & & \\ & -8 & 0 & \\ 0 & & -5 & \\ & & & 0 \end{bmatrix} \qquad V_1 = (0, 0, 0, 1)$$

$$(A^*A - 4I) = \dots \qquad V_2 = (0, 0, 1, 0)$$

$$\dots \qquad V_3 = (0, 1, 0, 0)$$

$$V_4 = (1, 0, 0, 0)$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

rang $A = 3$

$$u_i = \frac{1}{\sigma_i} A v_i \qquad i \in \{2, 3, 4\}$$

$$u_1 = \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \dots = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_3 = \dots = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

dopolniter do ONB: ... (gram-schidt)

$$u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

N

Let A realna $m \times n$ matrika in K enotska krogla v $\mathbb{R}^n$.

Pokaži: množica $\{Av;\ v \in K\}$ je elipsoid v podprostoru $\operatorname{Im} A \subseteq \mathbb{R}^m$. Polosi elipsoida so ravno neničelne singularne vrednosti A.

$A = U \Sigma V^*$ je SVD za A.

Let poljuben $v \in K$ razvijemo ga po stolpcih V

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

velja $\sqrt{\alpha_1^2 + \dots + \alpha_n^2} \leq 1$, ker je v v tej krogli z radijem 1.

$$\forall i \in \{1, \dots, \operatorname{rang} A\}: \ w_i := \frac{1}{\sigma_i} A v_i$$

$$A v_i = \sigma_i u_i$$

Preslikamo v čez A:

$$Av = \alpha_1 A v_1 + \dots + \alpha_n A v_n =$$

$$\underbrace{\boxed{\alpha_1 \sigma_1}}_{y_1} u_1 + \dots + \underbrace{\boxed{\alpha_r \sigma_r}}_{y_r} u_r + \underset{r+1}{0} + \dots + \underset{n}{0}$$

$$\frac{y_1^2}{\sigma_1^2} + \dots + \frac{y_r^2}{\sigma_r^2} = \alpha_1^2 + \dots + \alpha_r^2 \leq \alpha_1^2 + \dots + \alpha_n^2 \leq 1$$

$$\Rightarrow \quad \alpha_1^2 + \dots + \alpha_r^2 \leq 1$$

sedaj vemo: $\forall v$: Av leži v r-dimenzionalnem elipsoidu.

dokažimo še, da $\operatorname{Im} A$ je ta r-dimenzionalen elipsoid.

sedaj vzemimo poljuben element iz elipsoida in najdimo v, da bo Av ta element.

$$\beta_1 u_1 + \dots + \beta_r u_r = w.$$

$$\frac{\beta_1}{\sigma_1} v_1 + \dots + \frac{\beta_r}{\sigma_r} v_r \stackrel{A}{\longmapsto} w$$