Poišči ortogonalni komplement:

$$V = \mathrm{Lin}\{\overset{u_1}{(1,-2,0,1,0)}, \overset{u_2}{(0,1,0,1,1)}, \overset{u_3}{(3,-1,2,0,-1)}\} \text{ v } \mathbb{R}^5$$

v stand. skal. prod.

1. Poiščemo OB za V: Gram-Schmidt:

$$v_1 = (1,-2,0,1,0)$$

$$v_2 = (0,1,0,1,1) - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2 = 6}(1,-2,0,1,0) =$$

$$= (0,1,0,1,1) - \frac{-1}{6}(1,-2,0,1,0) =$$

$$= \left(\frac{-1}{6}, \frac{4}{6}, 0, \frac{7}{6}, 1\right)$$

$$v_3 = (3,-1,2,0,-1) - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2}\left(\frac{-1}{6}, \frac{2}{6}, 0, \frac{7}{6}, 1\right)$$

$$- \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 =$$

$$= \left(\frac{38}{17}, \frac{16}{17}, 2, \frac{-16}{17}, \frac{-10}{17}\right)$$

Sedaj moramo dopolniti v_1, v_2, v_3 do baze $\mathbb{R}^5$ z 2 vektorjema v_4, v_5. Gotovo bosta ustrezna dopolnitev dva vektorja iz standardne baze.

Uganemo e_3 in e_5. Če bi ugani narobe bi po gram schmidtu dobili 0.

$$v_4 = e_3 - \cancel{\frac{\langle e_3, v_1\rangle}{\|v_1\|^2}}^{\,0} v_1 - \cancel{\frac{\langle e_3, v_2\rangle}{\|v_2\|^2}}^{\,0} v_2 - \frac{\langle e_3, v_3\rangle}{\|v_3\|^2} v_3 =$$

$$= (0,0,1,0,0) - \frac{2}{\|v_3\|^2}\left(\frac{38}{17}, \frac{16}{17}, 2, \frac{-16}{17}, \frac{-10}{17}\right) = \dots$$

$$v_5 = e_5 - \frac{\langle e_5, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle e_5, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle e_5, v_3\rangle}{\|v_3\|^2} v_3 - \frac{\langle e_5, v_4\rangle}{\|v_4\|^2} = \dots$$

$$V^\perp = \mathrm{Lin}\{v_4, v_5\}$$

Naj bodo V_i vektorski podprostori od vpssp W.

a.) $\underline{S \subseteq W \Rightarrow (S^\perp)^\perp = \mathrm{Lin}\{S\}}$

pod MNOŽICA, nenujno PODPROSTOR

$$S^\perp = (\mathrm{Lin}\{S\})^\perp$$

$$(S^\perp)^\perp = \underbrace{((\mathrm{Lin}\{S\})^\perp)^\perp}_{} = \mathrm{Lin}\{S\}$$

to pa je vektorski prostor, zato velja $V^{\perp\perp} = V$.

b.) $\underline{V_1 \subseteq V_2 \Rightarrow V_2^\perp \subseteq V_1^\perp}$

Predpostavimo $V_1 \subseteq V_2$.

$(\Longrightarrow)$:

$\underline{V_2^\perp \subseteq V_1^\perp}$

vzemimo poljuben $x \in V_2^\perp \Rightarrow \forall v_2 \in V_2 : \langle v_2, x\rangle = 0$.

dokažimo $\underline{x \in V_1^\perp}$: $\forall v_1 \in V_1 : \langle v_1, x\rangle = 0$.

je očitno, saj je $V_1 \subseteq V_2$ po predpostavki.

(*)

c.) $\underline{(V_1 + V_2)^\perp = (V_1)^\perp \cup (V_2)^\perp}$

$x \in (V_1 + V_2)^\perp \Longleftrightarrow \forall v_1 \in V_1, v_2 \in V_2 : \langle x, v_1 + v_2\rangle = 0 = \langle x, v_1\rangle + \langle x, v_2\rangle$

$\hookrightarrow$ torej tudi v posebnih primerih: $v_1 = 0$ in $v_2 = 0$, ko velja $0 = 0 + \langle x, v_2\rangle$.

$\langle x, v_1\rangle \overset{?}{=} 0$, $\langle x, v_2\rangle \overset{?}{=} 0$

$\iff x \in V_1^\perp, \; x \in V_2^\perp$

$\iff (\#)$

d) $(V_1 \cap V_2)^\perp = (V_1)^\perp + (V_2)^\perp$

let: U_1, U_2 : $U_1 = (V_1)^\perp$

$U_2 = (V_2)^\perp$

$(U_1 + U_2)^\perp = (U_1)^\perp \cap (U_2)^\perp$

$(V_1^\perp + V_2^\perp)^\perp = V_1^{\perp\perp} \cap V_2^{\perp\perp} = V_1 \cap V_2$

Dokaži: $\underline{B^* A = 0} \iff \underline{\operatorname{Im} B \subseteq (\operatorname{Im} A)^\perp}$

$(\Rightarrow)$: $B^* A = 0 \iff \forall x \in V: B^* A x = 0 \iff$

$\forall x, y \in V: \langle B^* A x, y \rangle = 0 \iff$

$\langle \underbrace{Ax}_{v}, \underbrace{By}_{w} \rangle = 0$

$\iff \forall v, w \in \operatorname{Im} A: \langle v, w \rangle = 0 \sim v \perp w$

$\iff \operatorname{Im} A \perp \operatorname{Im} B \iff$

$\operatorname{Im} A \subseteq (\operatorname{Im} B)^\perp$ ali

$\operatorname{Im} B \subseteq (\operatorname{Im} A)^\perp$ BSŠ.

da velja $\operatorname{Im} B \subseteq (\operatorname{Im} A)^\perp$

N

Ali je $A=\begin{bmatrix}0&0&i\\0&1&0\\i&0&0\end{bmatrix}$ sebiadjungirana, normalna, unitarna?

$A^* = A$ NE

$A^*A = AA^*$ JA

JA

$A^* = A^{-1}$

$\Leftrightarrow$

$AA^* = A^*A = I$

$\overline{A}^T$

$//$

$A^* = \begin{bmatrix}0&0&-i\\0&1&0\\-i&0&0\end{bmatrix} \neq A \Rightarrow$ ni sebiadjungirana.

$A^*A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \Longrightarrow$ je unitarna $\Rightarrow$ je normalna

N

$a := [a_1 \;\; a_2 \;\; \dots \;\; a_n]^T \in \mathbb{C}^n$

$a^* = [\overline{a_1} \;\; \overline{a_2} \;\; \dots \;\; \overline{a_n}]$

$\|a\| = 1$ $(*)$

$A := I - \alpha a a^*$; $\alpha \in \mathbb{C}$

Ugotovi, za katere α je A unitarna?

$\hookrightarrow AA^* = I$

$A^* = (I - \alpha a a^*)^* = I - \overline{\alpha a a^*}^T =$

$= I - \overline{\alpha} (\overline{a}^*)^T \overline{a}^T = I - \overline{\alpha}\, \overline{\overline{a}^T}^T a^* = I - \overline{\alpha} a a^*$

$AA^* = (I - \alpha a a^*)(I - \overline{\alpha} a a^*) = I - \overline{\alpha} a a^* - \alpha a a^* + \alpha \overline{\alpha} a a^* a a^*$

$\overline{a_1} a_1 + \dots + \overline{a_n} a_n = \langle a, a \rangle = 1$ $(*)$

...

$= AA^*$

Glej sliko table

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N

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Obravnavaj antihermitske matrike

oblike $Q^H = -Q$

1.) Ali so normalne?

$$Q Q^H = -Q Q = Q Q^H \checkmark \text{ so.}$$

2.) Kakšne so njene lastne vr?

$$Qv = \lambda v$$

(1) $v^H Q v = ((v^H Q)^H)^H v = (-Qv)^H v = (-\lambda v)^H v =$

$= -\bar{\lambda} v^H v$ $\qquad -\bar{\lambda} = \lambda$

$\hookrightarrow \operatorname{Re} \lambda = 0$

(2) $v^H Q v = v^H \lambda v = \lambda v^H v$

N

Pokaži, da lahko vsako kompleksno matriko zapišemo kot vsoto hermitske in antihermitske.

Let A poljubna kompleksna matrika:

$$A = \underbrace{\frac{A + A^H}{2}}_{\text{let } H_1 = \operatorname{Re}(A)} + i \underbrace{\frac{A - A^H}{2i}}_{i \operatorname{Im}(A) = \text{let } H_2}$$

dokazati je treba, da je hermitska in antihermitska.

H_1 hermitska

$$H_1^H = \left(\frac{A+A^H}{2}\right)^H = \frac{A^H+A}{2} = \frac{A+A^H}{2} = H_1$$

H_2 antihermitska, t.j. $H_2^H = -H_2$

$$iH_2^H = \left(i\frac{A-A^H}{2i}\right)^H = -i\frac{A^H-A}{-2i} = i\frac{A^H-A}{2i} =$$

$$= -i\frac{A-A^H}{2i} = -iH_2 \checkmark$$

Naj $A = [a_{ij}]$ hermitska matrika in λ njena največja lastna vr.

Pokaži $\forall i: \lambda \geq a_{ii}$ (λ je kvečjemu največji diagonalec).

Namig: $\frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \lambda$

hermitska $\Rightarrow$ ima ONB diagonalizacijo $= \{b_1, \dots b_n\}$

$$Ab_i = \lambda_i b_i$$

$$x = \sum_{i=1}^{n} \alpha_i b_i$$

$\hookrightarrow$ razvit po bazi

$$\langle Ax, x\rangle = \left\langle A\sum_{i=1}^{n}\alpha_i b_i, \sum_{i=1}^{n}\alpha_i b_i\right\rangle =$$

$$= \left\langle \sum_{i=1}^{n}\alpha_i Ab_i, \sum_{i=1}^{n}\alpha_i b_i\right\rangle = \left\langle \sum_{i=1}^{n}\alpha_i\lambda_i b_i, \sum_{i=1}^{n}\alpha_i b_i\right\rangle =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \alpha_i \overline{\alpha_j} \underbrace{\langle b_i, b_j \rangle}_{\text{dirac delta}} \leq \sum_{i=1}^{n} \lambda a_i \overline{\alpha_i}$$

$$\delta(i-j) = \begin{cases} 1 ; & x = 0 \\ 0 ; & x \neq 0 \end{cases}$$

$$\langle x, x \rangle = \langle \sum_{i=1}^{n} \alpha_i b_i , \sum_{j=1}^{n} \alpha_j b_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \underbrace{\langle b_i, b_j \rangle}_{\delta(i-j)} =$$

$$= \sum_{i=1}^{n} \alpha_i \overline{\alpha_i}$$

$$\frac{\langle Ax, x \rangle}{\langle x, x \rangle} \leq \frac{\lambda \sum_{i=1}^{n} \alpha_i \overline{\alpha_i}}{\sum_{i=1}^{n} \alpha_i \overline{\alpha_i}} = \lambda$$

za $x = e_i$ → element std baze:

$$\frac{\cancel{\langle Ae_i, e_i \rangle} \to a_{ii}}{\cancel{\langle e_i, e_i \rangle} \to 1} = a_{ii}$$

N

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$$ poišči ortogonalno P, da bo $P^T A P = D$

P ortogonalna $\Leftrightarrow$ stolpci tvorijo ONB

A simetrična $\Rightarrow$ A normalna $\Rightarrow$ da se jo diagonalizirati, tako so lastni vektorji ONB.

$$\Delta_A(\lambda) = -(\lambda - 6)(\lambda - 3)^2$$

$$(A - 6I) = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$x_2 + x_3 = 0$

$x_1 - x_2 - 2x_3 = 0$

...

$x_1 = x_3$

vemo, da je ortog. na V_1

$V_1 = (-1, 1, -1)$

$V_2 = (0, 1, 1)$

$V_3 = (-1, 1, -1) \times (0, 1, 1) =$

$= (2, 1, -1)$

$V_1' = \frac{V_1}{\sqrt{3}}$

$V_2' = \frac{V_2}{\sqrt{2}}$

$V_3' = \frac{V_3}{\sqrt{6}}$

$P = \begin{bmatrix} V_1' & V_2' & V_3' \end{bmatrix}$ $\quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Naloga

Pokaži, da je produkt ortogonalnih/unitarnih matrik ortogonalna/unitarna matrika.

Dokaz je isti, naredimo za unitarnost.

Q, R sta ortogonalni:

$Q^T Q = Q Q^T = I \quad$ in $\quad R^T R = R R^T = I$

$(QR)^T QR = R^T Q^T Q R = R^T I R = I$