N

$A: \mathbb{R}^4 \to \mathbb{R}^3$ linearna v standardnih bazah

$$A_s = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ 2 & 2 & 2 & 4 \end{bmatrix}$$

Določi bazo jedra in slike v standardnih bazah po Gaussu.

$\operatorname{Ker} A = \{ x \in \mathbb{R}^4 ; \quad Ax = 0 \}$

$\operatorname{Im} A = \{ y \in \mathbb{R}^3 ; \ \exists x \in \mathbb{R}^4 \text{ t: } Ax = y \}$

$$\begin{bmatrix} 1 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ 2 & 2 & 2 & 4 \end{bmatrix} \xrightarrow[\text{II}-=\text{I}]{} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -1 & 0 & -1 \\ 2 & 2 & 2 & 4 \end{bmatrix} \xrightarrow[\text{III}-=2\text{I}]{} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & \cancel{-1}\ 1 & 0 & \cancel{-1}\ 1 \\ 0 & -4 & 0 & -4 \end{bmatrix} \xrightarrow[\text{III}+=4\text{II}]{} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{II} += -\frac{1}{2}\text{III}.$

$x_1 + 3x_2 + x_3 + 4x_4 = 0$

$x_2 + x_4 = 0 \qquad x_2 = -x_4$

$0 = 0$

$x_1 - 3x_4 + x_3 + 4x_4 = 0$

$x_1 = -x_3 - x_4$

$\dim \operatorname{Ker} A = 2$

$\operatorname{Ker} A = \{ (-x_3 - x_4, -x_4, x_3, x_4) ; \ x_3, x_4 \in \mathbb{R} \}$

$B_{\operatorname{Ker} A} = \{ (-1, -1, 0, 1), \ (-1, 0, 1, 0) \}$

$\operatorname{Im} A = \operatorname{Lin}\{ (1,1,2), (3,2,2), \cancel{(1,1,2)}, (4,3,4) \}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 2 \\ 4 & 3 & 4 \end{bmatrix} \xrightarrow[\text{III}-=4\text{I}]{} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 2 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow[\text{II}-=3\text{I}]{} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$B_{\operatorname{Im} A} = \{ (1,1,2), (0,1,4) \}$

{dimenzijska enačba

2. $A: \mathbb{R}^2 \to \mathbb{R}^3$ linearna za katero velja

$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ a) izračunaj $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$(1,0) = \frac{1}{2}(1,1) - \frac{1}{2}(-1,1)$

$A(1,0) = A\left(\frac{1}{2}(1,1) - \frac{1}{2}(-1,1)\right) =$

$= \frac{1}{2} A(1,1) - \frac{1}{2} A(-1,1) = (0, 1/2, 1) - (1, 1/2, 0) =$

$= (-1, 0, 1)$

b.) izračunaj $A(x,y)$

$A(x,y) = x A(1,0) + y A(0,1)$

$A(0,1) = A(1,1) - A(1,0) = (0,1,2) - (-1,0,1) =$

$= (1, 1, 1)$

$A(x,y) = x(-1,0,1) + y(1,1,1) = (-x, 0, x) + (y, y, y) =$
$(y-x, y, y+x)$

c.) zapiši matriko A v standardnih bazah.

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

d.) določi bazo za $\operatorname{Im} A$ in $\ker A$

$\ker A = \{(0,0)\}$ $\quad B_{\ker A} = \emptyset$

$$\underbrace{\dim \operatorname{Im} A}_{} + \underset{\substack{\| \\ 0}}{\dim \ker A} = \underset{\substack{\| \\ 2}}{\dim \mathbb{R}^2}$$

$B_{\operatorname{Im} A} = \{(0,1,2), (2,1,0)\}$

N

$F: \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ $\quad f(x) = sl(x) = x_{11} + x_{22} + \dots + x_{nn}$

↳ sum diagonalnih el.

dokaži, da je F linearna preslikava.

$F(aX + bY) = aFX + bFY$

$$F(aX+bY) = F\left(\begin{bmatrix} ax_{11} & \dots & ax_{1n} \\ \vdots & \ddots & \vdots \\ ax_{n1} & \dots & ax_{nn} \end{bmatrix} + \begin{bmatrix} by_{11} & \dots & by_{1n} \\ \vdots & \ddots & \vdots \\ by_{n1} & \dots & by_{nn} \end{bmatrix}\right) =$$

$= a(x_{11} + \dots + x_{nn}) + a(y_{11} + \dots + y_{nn}) =$

$= aFX + bFY$

b.) za $n=2,3$ zapiši matriko preslikave v standardnih bazah.

$$n=2: \left\{ \overset{\text{"1"}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{\text{"2"}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{\text{"3"}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{\text{"4"}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$1 \qquad 0 \qquad 0 \qquad 1$$

matrika presl.: $[1\ 0\ 0\ 1]$

$$n=3: \left\{ \overset{\text{"1"}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \dots, \overset{\text{"9"}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \right\}$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$1 \qquad \dots \qquad 1$$

matrika presl.: $[1, 0, 0, 0, 1, 0, 0, 0, 1]$.

c)

za $n=2,3$ določi katero bazo $\ker F$ in $\operatorname{Im} F$

$n=2$: $\operatorname{Im} F = \mathbb{R}$ $\qquad B_{\operatorname{Im} F} = \{ 1 \}$

$$\dim \ker F + \underbrace{\dim \operatorname{Im} F}_{1} = \underbrace{\dim \mathbb{R}^{2\times 2}}_{4}$$

$$\Rightarrow \dim \ker F = 3$$

$$B_{\ker F} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$n=3$: $\operatorname{Im} F = \mathbb{R}$ $\qquad B_{\operatorname{Im} F} = \{1\}$

$$\underbrace{\dim \ker F}_{8} + \underbrace{\dim \operatorname{Im} F}_{1} = \underbrace{\dim \mathbb{R}^{3\times 3}}_{9}$$

$$B_{\ker F} = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{bmatrix} \right\}$$

N

Dokaži: $\operatorname{rang}(A) = \operatorname{rang}(A^T)$

$\operatorname{rang}(A) := \dim \operatorname{Im} A$

F je poljubno polje

let $A \in M_{m\times n}(F)$

$\operatorname{rang} A = r$

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q$$

odmislimo

$$A^T = Q^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}^T P^T$$

$$\operatorname{rang} A^T = \operatorname{rang}\left(Q^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P^T\right)$$

vrstično/stolpično ekvivalentne matrike imajo isti rang (trditev s predavanj)

$$= \operatorname{rang}\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P^T\right) = \operatorname{rang}\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right) = r = \operatorname{rang} A$$

N

Določi rang $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 3 & 4 \\ 3 & -5 & 8 & 7 \end{bmatrix} \xrightarrow[\text{III} -= \text{I}\cdot 3]{} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 3 & 4 \\ 0 & -14 & 14 & 4 \end{bmatrix} \xrightarrow[\text{II} -= \text{I}\cdot 2]{} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -7 & 7 & 2 \\ 0 & -14 & 14 & 4 \end{bmatrix}$

$\operatorname{rang} A = \dim \operatorname{Im} A = 2$ $\qquad \xrightarrow[\text{III} -= 2\text{II}]{} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -7 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

N

Dokaz Kronecker–Capellijev izrek:

sistem $Ax = b$ je rešljiv $\Leftrightarrow \operatorname{rang}[A|b] = \operatorname{rang} A$

$\Updownarrow$

$b \in \operatorname{Im} A$

$\operatorname{Im}[A|b] = \operatorname{Lin}\{A^1, A^2, \ldots, A^n, b\}$

$\operatorname{Im} A = \operatorname{Lin}\{A^1, A^2, \ldots, A^n\}$

$\Leftrightarrow \operatorname{Im} A = \operatorname{Im}(A|b) \Leftrightarrow \dim \operatorname{Im} A = \dim \operatorname{Im}[A|b]$

$\quad$ N $\quad \text{rang } A = \text{rang}[A|b].$

N

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

P, Q obrnljiva

$r = \text{rang } A$

želimo

$$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} I_r & Q \\ P & 0 \end{bmatrix}$$

$$\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} PA & P \\ I & 0 \end{bmatrix}$$

$$\begin{bmatrix} PA & P \\ I & 0 \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} PAQ & P \\ Q & 0 \end{bmatrix}$$

z gaussom:

$$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} PA & P \\ I & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} PAQ & P \\ Q & 0 \end{bmatrix} \longrightarrow \dots$$

N

Poišči P, Q, za katere je $P\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 4 & 5 & 6 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \dots \longrightarrow \begin{bmatrix} \boxed{I_r} & P \\ Q & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 4 & 5 & 6 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[II -= 2I]{} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \dots \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} & 2/3 \\ 0 & 1 & 0 & 4/3 & -1/3 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

P

Q

+ preizkus.

N

$\text{rang } AB \leq \min\{\text{rang } A, \text{rang } B\}$ $\qquad B: U \to V$, $A: V \to W$

$\underline{rAB \leq rA}$ $\wedge$ $\underline{rAB \leq rB}$

$\operatorname{Im} AB = \{ABu,\ u \in U\} = \{Ax;\ x = Bu,\ u \in U\} =$

$= \{Ax;\ x \in \operatorname{Im} B\} \leq \{Ax;\ x \in V\}$

Iz $\operatorname{Im} AB \subset \operatorname{Im} A$ sledi $rAB \leq rA$

$\underline{rAB \leq rB}$

$rAB = r(AB)^T = r\,B^T A^T \leq rB^T = rB \ \square$