N

$v_1 = (2,3,-1,4)$ $v_2 = (1,2,1,2)$ $v_3 = (4,7,1,8)$

$v_4 = (1,1,-2,2)$

a so linearno neodvisni?

poišči bazo $\mathrm{Lin}\{v_1, \dots v_4\}$.

$$\begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & 2 & 1 & 2 \\ 4 & 7 & 1 & 8 \\ 1 & 1 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & 7 & 1 & 8 \\ 2 & 3 & -1 & 4 \end{bmatrix} \xrightarrow[\text{IV}-2\text{I}]{\text{II}-=\text{I},\ \text{III}-=2\text{I}} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & +3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \xrightarrow[\text{IV}-=\text{II}]{} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & +3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{III}-=\text{II}} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

niso LN, ker matrika ni obrnljiva.

baza sta $u_1 = (1,1,-2,2)$

$u_2 = (0,1,3,0)$

N

$U = \{p \in \mathbb{R}_3[x] ; p(x) = p(-x)\}$ poišči bazo.

Baza od $\mathbb{R}_3[x]$: $x^0, x^1, x^2, x^3$

presek sodih polinomov in polinomov stopnje 3.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$p(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3$$

$p(x) = p(-x) \iff a_1 = a_3 = 0$

naš $p(x)$ je oblike $p(x) = a_0 + a_2 x^2$

baza je: $1, x^2$.

N

$Q = \{p \in \mathbb{R}[x] ; p(1) = 0\}$

$S = \{p \in \mathbb{R}[x] ; p(2) = 0\}$

$Q_n = Q \cap \mathbb{R}_n[x]$

$S_n = S \cap \mathbb{R}_n[x]$

1. Določi $Q + S \subseteq \mathbb{R}[x]$

ideja $Q + S = \mathbb{R}[x]$. poljuben polinom moramo napisati kot vsoto polinomov iz $Q$ in $S$:

$p(x)$ $q(x)$ $s(x)$

$p(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots$

$(x-1)$ in $(x-2)$ sta tuja

$\gcd((x-1),(x-2)) = 1$

$q(x) = (x-1)\tilde{q}(x)$

$s(x) = (x-2)\tilde{s}(x)$

$(x-1)a(x) + (x-2)b(x) = 1 \quad / \cdot p(x)$

$$\underbrace{(x-1)a(x)p(x)}_{\in Q} + \underbrace{(x-2)b(x)p(x)}_{\in S} = p(x)$$

2. $Q \cap S = \{ p \in \mathbb{R}[x] \; ; \; p(1) = p(2) = 0 \}$

3. BAZA za $Q_n$:

$$q(x) = (x-1)\tilde{q}(x)$$

$$\text{stopnja}(\tilde{q}(x)) \leq n-1$$

$$\tilde{q}(x) \in \mathbb{R}_{n-1}[x]$$

$\mathbb{R}_n[x]$ pomeni polinome stopnje $n$ in nižjih stopenj!

baza $Q_n$:

$$B_{Q_n} = \{ 1(x-1), x(x-1), x^2(x-1), \ldots, x^{n-1}(x-1) \}$$

a je baza LN?

predp: $\sum_{i=0}^{n-1} \alpha_i x^i (x-1) = 0$

$$= \sum_{i=0}^{n-1} \alpha_i x^{i+1} - \sum_{i=0}^{n-1} \alpha_i x^i = x^i \left( x \sum_{i=0}^{n-1} \alpha_i - \sum_{i=0}^{n-1} \alpha_i \right) = x^i \sum_{i=0}^{n-1} \alpha_i (x-1) =$$

$$= x \sum_{i=0}^{n-1} \alpha_i x^i - \sum_{i=0}^{n-1} \alpha_i x^i = \underbrace{\left( \sum_{i=0}^{n-1} \alpha_i x_i \right)}_{\substack{\text{Baza za } \mathbb{R}_{n-1}[x] \\ \Downarrow \\ \text{ničelni polinom}}} \underbrace{(x-1)}_{\text{ni ničelni polinom}}$$

$$\Downarrow$$

$\vec{\alpha} = \vec{0}$.

torej je ta baza res LN in zato res baza

da je, ogrodje je očitno.

4. baza $S_n$?

$$s(x) = (x-2)\tilde{s}(x) \quad ; \quad \text{stopnja}(\tilde{s}(x)) \leq n-1$$

$$B_{S_n} = \{ (x-2) \cdot 1, (x-2) \cdot x, (x-2) \cdot x^2, \ldots, (x-2) \cdot x^{n-1} \}$$

dokazali bi enako.

5. baza $S_n \cap Q_n$?

$$\Downarrow$$

$$p(x)$$

$$p(x) = (x-1)(x-2)\tilde{p}(x) \qquad \text{stopnja}(\tilde{p}(x)) \leq n-2$$

$$B_{S_n \cap Q_n} = \{ (x-1)(x-2)1, (x-1)(x-2)x, \ldots (x-1)(x-2)x^{n-2} \}$$

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N

$$X = \text{Lin}\{ (5, 2, -11, 9), (-4, 3, 18, 2) \}$$

$$Y_t = \text{Lin}\{ (1, 3, 3, 7), (2, -1, -8, 0), (2, -3, t, -4) \}$$

a: za katere $t \in \mathbb{R}$ je $X \subseteq Y_t$?

b: Ali $\exists t \in \mathbb{R}$: $X = Y_t$?

iščimo bazi:

$$\begin{bmatrix} 5 & 2 & -11 & 9 \\ -4 & 3 & 18 & 2 \end{bmatrix} \xrightarrow[I+=II]{} \begin{bmatrix} 1 & 5 & 7 & 11 \\ -4 & 3 & 18 & 2 \end{bmatrix} \xrightarrow[II+=4I]{} \begin{bmatrix} 1 & 5 & 7 & 11 \\ 0 & 23 & 46 & 46 \end{bmatrix}_{II/=23} \to \begin{bmatrix} 1 & 5 & 7 & 11 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[II+=I]{} \begin{bmatrix} 5 & 2 & -11 & 9 \\ 1 & 5 & 7 & 11 \end{bmatrix} \xrightarrow[I-=5II]{} \begin{bmatrix} 0 & -23 & -36 & -46 \\ 1 & 5 & 7 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 7 \\ 2 & -1 & -8 & 0 \\ 2 & -3 & t & -4 \end{bmatrix} \xrightarrow[III-=2I]{II-=2I} \begin{bmatrix} 1 & 3 & 3 & 7 \\ 0 & -7 & -14 & -14 \\ 0 & -9 & t-6 & -25 \end{bmatrix}_{II/=-7} \to \begin{bmatrix} 1 & 3 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & -9 & t-6 & -25 \end{bmatrix}$$

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N

Def: let $U_1 + \dots + U_n$ vsota v. podpr.

Pravimo, da je vsota direktna, če velja

$\forall u_1 \in U_1, \dots, u_n \in U_n \ \exists: u_1 + \dots + u_n = 0$

Izrek: karakterizacija direktnih vsot:

lot $U_1, \dots, U_n$ v. podprostori KRVP V.

NTSE (naslednje trditve so ekvivalentne)

1. $U_1 + \dots + U_n$ je direktna
2. $\forall v \in U_1 + \dots + U_n$ lahko na enoličen način zapišemo
 tot $v = \alpha_1 u_1 + \dots + \alpha_n u_n \quad u_i \in U_i$
3. $u_{11} \dots u_{1 r_1}$ je baza $U_1$ (razsežnost prvega prostora) $\Rightarrow u_{11} \dots u_{1r_1}, \dots, u_{n1} \dots u_{nr_n}$
 $\vdots$ je baza $U_1 + \dots + U_n$
 $u_{n1} \dots u_{nr_n}$ je baza $U_2$
4. $\dim(U_1 + \dots + U_n) = \dim U_1 + \dots + \dim U_n$
5. $\forall i = 1, \dots, n-1$ velja $(U_1 + \dots + U_i) \cap (U_{i+1}) = \{0\}$

Dokaz $1 \Rightarrow 2$: $U_1 + \dots + U_n = \{u_1 + \dots + u_n;\ u_1 \in U_1, \dots, u_n \in U_n\}$

recimo, da obstajata dva zapisa vektorja v.

$v = u_1 + \dots + u_n = w_1 + \dots + w_n$

$u_1 + \dots + u_n - (w_1 + \dots + w_n) = 0$

$(u_1 - w_1) + (u_2 - w_2) + \dots + (u_n - w_n) = 0$

baje nekako sledi

$u_1 = w_1$
$\vdots$ ? ampak okej, verjamem.
$u_n = w_n$

$2 \Rightarrow 3$: $u \in U_1 + \dots + U_n$

$u = u_1 + \dots + u_n$

$\forall u_i$ lahko izrazimo tot LK baze $u_{i1} \dots u_{ir_i}$

sledi B je ogrodje

Dokažimo B je LN.

$\underbrace{\alpha_{11} u_{11} + \dots + \alpha_{1r_1} u_{1r_1}}_{u_1 \in U_1} + \dots + \underbrace{\alpha_{n1} u_{n1} + \dots + \alpha_{nr_n} u_{nr_n}}_{u_n \in U_n} = 0$

Po enoličnosti zapisa $u_1 = u_2 = \dots = u_n$

ker so $u_{11}, \dots, u_{1r_1}$ baza $U_1 \Rightarrow \alpha_{11} = \dots = \alpha_{1r_1} = 0$

$\vdots$

$u_{n1}, \dots, u_{nr_n}$ baza $U_n \Rightarrow \alpha_{n1} = \dots = \alpha_{1r_n} = 0$

$$\vec{\alpha} = \vec{0}$$

$3 \Rightarrow 4$: sledimo iz definicije dimenzije: dim je moč baze.

$4 \Rightarrow 5$: s pomočjo dimenzijske formule:

$$\dim(U_1+U_2) \leq \dim(U_1) + \dim(U_2)$$

$$\dim(U_1+U_2) = \dim(U_1) + \dim(U_2) \Leftrightarrow U_1 \cap U_2 = \{0\}$$

$$\dim(U_1 + \dots + U_n) \overset{\{0\}\text{, zato je} \leq \text{enakost}}{\leq} \dim U_n + \dim(U_1 + \dots + U_{n-1}) \overset{=\{0\}\text{, zato je} \leq \text{enakost}}{\leq} \dim U_n + \dim U_{n-1} + \dim(U_1 + \dots + U_{n-2})$$

$$\leq \dim U_n + \dim U_{n-1} + \dots + \dim U_1$$

$\cap = \{0\}$ zato je $\leq$ enakost.

$5 \Rightarrow 1$

$$u_1 + \dots + u_n \overset{?}{=} 0$$

$u_1 + \dots + u_{n-1} = -u_n$ vzemimo $i = n-1$

$\in U_1 + U_2 + \dots + U_{n-1}$, torej $-u_n = 0 = u_n$

in $u_1 + \dots + u_{n-1} = 0$

vzemimo $i = n-2$

itd

indukcija na $n$ očitno

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N

Določi bazo za $U = \{x \in \mathbb{R}^4 ; x_1 + x_2 + x_3 + x_4 = 0\}$

$V = \{x \in \mathbb{R}^4 ; x_2 = x_3 = x_4 = 0\}$

Prepričaj se, da je $\mathbb{R}^4 = U + V$ direktna.

$$B_V = \{(1,0,0,0)\}$$

$$B_U = \{(0,0,1,-1), (0,1,0,-1), (1,0,0,-1)\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[\text{IV} - \text{I}]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \dots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

polna matrika $\Rightarrow \mathbb{R}^4 = U + V$

da je $U+V$ direktna, je $U \cap V = \{0\}$.

$a \in V$ : $a = (\alpha, 0, 0, 0)$

$a \in V \cap U$ : $a = (\alpha, 0, 0, 0)$ $\Leftarrow$ pogoj $V$

$\alpha + 0 + 0 + 0 = 0$ $\Leftarrow$ pogoj $U$

$\Rightarrow \alpha = 0$

edini tak je $a = (0,0,0,0)$

$U + V = \{0\}$ ✓.