Uporaba ortogonalnih baz.

Baza je ortogonalna, če so njeni elementi paroma ortogonalni.
(= ortogonalna množica, ki je ogrodje)

opomba: LN sledi iz ortogonalnosti in neničelnosti.

Baza je ortonormirana, če je ortogonalna in normirana.

[FOURIEROV RAZVOJ]

let V KRUPSSP, $\{v_1, \dots, v_n\} = B$ ortogonalna baza za V in $v \in V$ poljuben element. Kako razvijemo v po B, vedoč, da je ta baza ortogonalna? lažje je.

Ker je B ogrodje, $\exists \alpha_1, \dots, \alpha_n \in F$: $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ $\quad / \cdot v_i$ (skalarno)

$$\langle v, v_i \rangle = \langle \alpha_1 v_1 + \dots + \alpha_n v_n, v_i \rangle \quad \text{linearnost } \langle \cdot, \cdot \rangle$$

$$\langle v, v_i \rangle = \alpha_1 \underbrace{\langle v_1, v_i \rangle}_{0} + \dots + \alpha_i \underbrace{\langle v_i, v_i \rangle}_{\neq 0} + \dots + \alpha_n \underbrace{\langle v_n, v_i \rangle}_{0} \quad \text{orto}$$

$$\langle v, v_i \rangle = \alpha_i \langle v_i, v_i \rangle$$

$$\alpha_i = \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$$

torej $\forall v \in V$ velja:

$$v = \sum_{i=1}^{n} \underbrace{\boxed{\frac{\langle v, v_i \rangle}{\underbrace{\langle v_i, v_i \rangle}_{\|v_i\|^2}}}}_{\text{fourierovi koeficienti}} v_i$$

fourierov razvoj v po B. opazimo eksplicitno formulo za koeficiente, saj imamo ortogonalno bazo

če pa imamo celo ortoNORMIRANO bazo, pa:

$$v = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\underbrace{\langle v_i, v_i \rangle}_{1}} v_i = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$$

[PARSEVALOVA IDENTITETA]

Radi bi izrazili $\|v\|$ s fourierovimi koeficienti za v.

let $v = \alpha_1 v_1 + \dots + \alpha_n v_n$; $\{v_1, \dots, v_n\}$ ortog. baza.

tedaj $\langle v, v \rangle = \langle \alpha_1 v_1 + \dots + \alpha_n v_n, \alpha_1 v_1 + \dots + \alpha_n v_n \rangle$

↓ linearnost 1. + konj. lin. 2.

$$= \alpha_1 \bar{\alpha}_1 \underbrace{\langle v_1, v_1 \rangle}_{\neq 0} + \dots + \alpha_1 \bar{\alpha}_n \underbrace{\langle v_1, v_n \rangle}_{0}$$

$$\vdots$$

$$\alpha_n \bar{\alpha}_1 \underbrace{\langle v_n, v_1 \rangle}_{0} + \dots + \alpha_n \bar{\alpha}_n \underbrace{\langle v_n, v_n \rangle}_{\neq 0} =$$

$$= \alpha_1 \bar{\alpha}_1 \langle v_1, v_1 \rangle + \dots + \alpha_n \bar{\alpha}_n \langle v_n, v_n \rangle =$$

$$= |\alpha_1|^2 \|v_1\|^2 + \dots + |\alpha_n|^2 \|v_n\|^2$$

vstavimo formule za α_i ↓

$$= \left| \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} \right|^2 \langle v_1, v_1 \rangle + \dots + \left| \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} \right|^2 \langle v_n, v_n \rangle$$

$$= \frac{|\langle v, v_1\rangle|^2}{\langle v_1, v_1\rangle^2} \cancel{\langle v_1, v_1\rangle} + \dots + \frac{|\langle v, v_n\rangle|^2}{\langle v_n, v_n\rangle^2} \cancel{\langle v_n, v_n\rangle} =$$

$$= \boxed{\|v\|^2 = \sum_{i=1}^{n} \frac{|\langle v, v_i\rangle|^2}{\langle v_i, v_i\rangle}}$$

PARSEVALOVA IDENTITETA

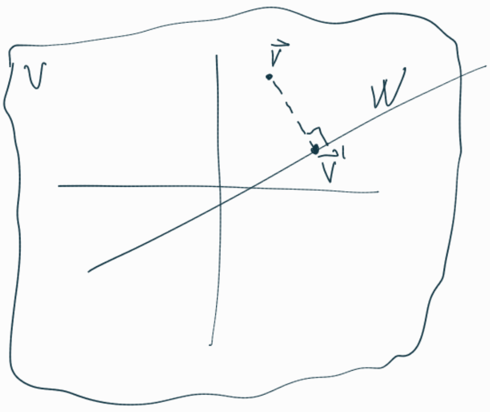
OPOMBA: za ortoNORMIRANO bazo velja:

$$\|v\|^2 = \sum_{i=1}^{n} \frac{|\langle v, v_i\rangle|^2}{\cancel{\langle v_i, v_i\rangle}} = \sum_{i=1}^{n} |\langle v, v_i\rangle|^2$$

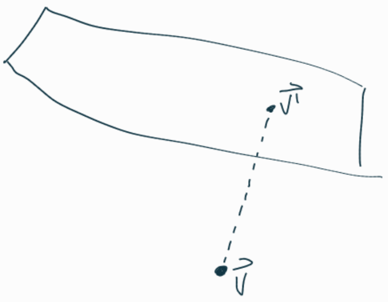
[PROJEKCIJA NA PODPROSTOR]

Let V KRVPSSP in W posprostor V

za vsak $v \in V$ želimo izračunati njegovo ortogonalno projekcijo na W.



ali pa



Definicija ortogonalne projekcije.

Vektor $\vec{v'} \in W$ je ortogonalna projekcija vektorja $v \in V$, če $\forall w \in W$: $\|v - v'\| \le \|v - w\|$

zdb v' je najbližje v od vseh elementov W.

Opomba: zadošča preveriti, da je $v - v'$ ortogonalen na vse elemente W. (pitagorov izrek)

$$\|v - w\|^2 = \|v - v' + \underbrace{v' - w}_{\in W}\|^2 = \|v - v'\|^2 + \underbrace{\|v' - w\|^2}_{\geq 0} \geq \|v - v'\|^2$$

\sun

po predpostavki je $v' - w \perp v - v'$

Let $\{w_1, \dots, w_k\}$ ortogonalna baza za W

$\Rightarrow$ imamo za ortog. proj. naslednjo formulo:

$$v' = \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \dots + \frac{\langle v, w_k\rangle}{\langle w_k, w_k\rangle} w_k =$$

kako dokažemo pitagorov izrek

let $a \perp b$.

$$\Rightarrow \|a+b\|^2 = \langle a+b, a+b\rangle = \langle a, a\rangle + \langle a, b\rangle \to 0 + \langle b, a\rangle \to 0 + \langle b, b\rangle =$$

$$= \|a\|^2 + \|b\|^2$$

$$v' = \sum_{i=1}^{k} \frac{\langle v, w_i\rangle}{\langle w_i, w_i\rangle} w_i$$

Radi bi ~~dokazal~~, da je

$$v - \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

pravokoten na vse elemente W.

uts: a je pravi podprostor vedno za dimenzijo manjši? JA

uts: kje smo pa dokazali, da res zadošča le preveriti pravokotnost? očitno

linearnost $\langle , \rangle$: zadošča preveriti pravokotnost na bazo W.

Vzemimo zdaj j med 1 in k.

$0 \Leftrightarrow i \neq j$

$$\left\langle v - \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i , w_j \right\rangle = \langle v, w_j \rangle - \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \langle w_i, w_j \rangle =$$

$$= \langle v, w_j \rangle - \frac{\langle v, w_j \rangle}{\cancel{\langle w_j, w_j \rangle}} \cancel{\langle w_j, w_j \rangle} = \langle v, w_j \rangle - \langle v, w_j \rangle = 0$$

$\rightarrow$ res pravokotno na bazo.

[OBSTOJ ORTOGONALNE BAZE]

$\hookrightarrow$ Gram-Schmidtova ortogonalizacija.

Radi bi dokazali, 1) da ima VSAK KRVPSSP ortogonalno bazo
2) vsako ortogonalno množico se da dopolniti do ortogonalne baze

konstruktiven dokaz (postopek: Gram-Schmidtova ortogonalizacija iz poljubne baze naredi ortogonalno).

let V kvpssp in $\{u_1, \dots, u_n\}$ njegova poljubna baza

let $v_1 = u_1$, $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$ $\quad u_2'$

$$v_3 = u_3 - \left(\frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \right) \quad u_3'$$

$$\vdots$$

$$v_n = u_n - \sum_{i=1}^{n-1} \frac{\langle u_n, v_i \rangle}{\langle v_i, v_i \rangle} v_i \quad u_n'$$

trdimo, da je $\{v_1, \dots, v_n\}$ ortogonalna baza za V.

opazimo, da je u_2' ortogonalna projekcija u_2 na $\mathrm{Lin}\{v_1\}$

u_3' — || — u_3 na $\mathrm{Lin}\{v_1, v_2\}$

u_n' — || — u_n na $\mathrm{Lin}\{v_1, \dots, v_{n-1}\}$

$$\Longrightarrow \begin{array}{l} \overbrace{u_2 - u_2'}^{v_2} \perp \mathrm{Lin}\{v_1\} \\ \underbrace{u_3 - u_3'}_{v_3} \perp \mathrm{Lin}\{v_1, v_2\} \\ \quad \cdots \\ \underbrace{u_n - u_n'}_{v_n} \perp \mathrm{Lin}\{v_1, \dots, v_{n-1}\} \end{array} \Rightarrow \begin{array}{l} v_2 \perp v_1 \\ v_3 \perp v_1, \ v_3 \perp v_2 \\ v_n \perp v_1, \dots, v_n \perp v_{n-1} \end{array}$$

$\Longrightarrow \{v_1, \dots, v_n\}$ paroma ortogonalni

dokazati je treba še neničelnost v_i.

v_1 je neničeln, ker $v_1 = u_1 \in$ Baza V

v_2 je neničeln, ker $v_2 = u_2 - \alpha v_1$,
$u_2 \neq \alpha v_1$, ker u_2 LN u_1

v_3 neničeln, ker $v_3 = u_3 - (\beta v_1 + \gamma v_2)$, ker
u_3, v_1, v_2 LN, torej $u_3 \neq \beta v_1 + \gamma v_2$

itd.

2) Kako ortogonalno množico dopolnemo do ortogonalne baze?

Let $\{u_1, \dots, u_k\}$ ortogonalna množica, torej je linearno neodvisna, torej jo lahko dopolnimo do baze. $\{u_{k+1}, \dots, u_n\}$ je dopolnitev do baze.

$\{u_1, \dots, u_k, u_{k+1}, \dots, u_n\}$ še ni ortogonalna.

Uporabimo Gram-Schmidtovo ortogonalizacijo. Opazimo, da je $v_1 = u_1$, $\underbrace{v_2 = u_2}$, $\dots$, $v_k = u_k$.

↳ ker je $\langle u_2, u_1 \rangle = 0$ po predpostavki

itd

torej nam $\{v_{k+1}, \dots, v_n\}$ dajo isto dopolnitev do ortogonalne baze

Primer GS ort.izacije iz analize:

Let $V = \mathbb{R}[x]$ stopnje ≤ 3 KRVPSSP

baza $u_1 = 1$, $u_3 = x^2$
$u_2 = x$, $u_4 = x^3$

skalarni produkt: $\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$

Konstruiraj pripadajočo ortogonalno bazo:

$\int_{-1}^{1} x \cdot 1\, dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0$

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = u_2 = x$

$\int_{-1}^{1} x^2 \cdot 1\, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - -\frac{1}{3} = \frac{2}{3}$

$\int_{-1}^{1} x^2 x\, dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

$v_3 = u_3 - \left(\frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \right) = x^2 - \frac{1}{3}$

$\int_{-1}^{1} 1 \cdot 1\, dx = x\Big|_{-1}^{1} = 1 - -1 = 2$

$v_4 = u_4 - \left(\frac{\overbrace{\langle u_4, v_1 \rangle}^{\dots = 0}}{\langle v_1, v_1 \rangle} v_1 + \frac{\overbrace{\langle u_4, v_2 \rangle}^{3/5}}{\langle v_2, v_2 \rangle} v_2 + \frac{\overbrace{\langle u_4, v_3 \rangle}^{\dots = 0}}{\langle v_3, v_3 \rangle} v_3 \right) =$

$= x^3 - \frac{3}{5} x$

sklep: $\{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x\}$ je ortogonalna baza za ta $(V, \langle \cdot, \cdot \rangle)$

$\|v_1\| = \sqrt{2}$, $\|v_2\| = \sqrt{2/3}$, $\|v_3\| = \sqrt{\frac{8}{45}}$, $\|v_4\| = \sqrt{\frac{8}{175}}$

normiranje bi sicer prineslo lepše formule, a bi v račune prineslo te grde konstante.

[ORTOGONALNI KOMPLEMENT]

Let V krvpssp in $S \subseteq V$, F polje V

Def.: ortogonalni komplement S je množica $S^\perp$; vsi vektorji iz V, ki so ortogonalni na S.

$S^\perp = \{ v \in V;\ \underbrace{\langle v, s \rangle = 0}_{v \perp s}\ \forall s \in S \} = \{ v \in V;\ v \perp S \}$

TRDITEV: $\forall S \subseteq V$: $S^\perp$ podprostor V

DOKAZ: $u_1, u_2 \in S^\perp$, $\alpha_1, \alpha_2 \in F \overset{?}{\Longrightarrow} \alpha_1 v_1 + \alpha_2 u_2 \in S^\perp$
↑ Def $S^\perp$

Def $S^\perp$: $\langle u_1, s\rangle = 0$ $\langle u_2, s\rangle = 0$ $\forall s \in S$ $\xrightarrow{\text{Lin1}}$ $\langle \alpha_1 u_1 + \alpha_2 u_2, s\rangle = \alpha_1 \underbrace{\langle u_1, s\rangle}_{=0} + \alpha_2 \underbrace{\langle u_2, s\rangle}_{=0} = 0$ $\forall s \in S$