JEDRO IN SLIKA LINEARNE PRESLIKAVE

Def.: Let U in V vektorska prostora nad istim poljem F
Let $L: U \to V$ linearna preslikava.

Definirajmo množici: $\underbrace{\text{Ker}}_{} L = \{u \in U \mid L(u) = 0\}$
↳ kernel = jedro = null space = $\mathcal{N}$

$\underbrace{\text{Im}}_{} L = \{L(u) \mid u \in U\}$
↳ Image = slika = range = $\mathcal{R}$ = zaloga vrednosti

TRDITVE:
1.) $\text{Ker } L$ je vektorski podprostor v U.
2.) $\text{Im } L$ je vektorski podprostor v V.

> vektorski podprostor:
> če vsebuje $\vec{a}, \vec{b}$, vsebuje tudi vse L.K. $\vec{a}, \vec{b}$.

DOKAZ: 1) $u_1, u_2 \in \text{Ker } L$ in $\alpha_1, \alpha_2 \in F \overset{?}{\Rightarrow} \alpha_1 u_1 + \alpha_2 u_2 \in \text{Ker } L$
↳ $L u_1 = 0$
↳ $L u_2 = 0$ $\Rightarrow \alpha_1 L u_1 + \alpha_2 L u_2 = 0$
ker je L linearna, je
$\Rightarrow L(\alpha_1 u_1 + \alpha_2 u_2) = 0$
po def. jedra sledi $\alpha_1 u_1 + \alpha_2 u_2 \in \text{Ker } L$.

2) $v_1, v_2 \in \text{Im } L$, $\beta_1, \beta_2 \in F \overset{?}{\Rightarrow} \beta_1 v_1 + \beta_2 v_2 \in \text{Im } L$
po def. slike:
$v_1 = L u_1$ za nek $u_1 \in U$
$v_2 = L u_2$ za nek $u_2 \in U$ $\Rightarrow$ $\beta_1 v_1 + \beta_2 v_2 = \beta_1 L u_1 + \beta_2 L u_2 \overset{\text{def Im}}{=} L(\underbrace{\beta_1 u_1 + \beta_2 u_2}_{\in U})$

def: $n(L) = \dim \text{Ker } L$ ↳ „nullity" = ničnost
$r(L) = \dim \text{Im } L$ ↳ „rank" = rang

opomba: Jedro in sliko smo definirali za lin. presl.
lahko ju definiramo tudi za matriko A, tako da je $U = F^n$ in $V = F^m$ za $A \in M_{m,n}(F)$.

tedaj
$$Au = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 a_{11} + \dots + u_n a_{1n} \\ \cdots \\ u_n a_{m1} + \dots + u_n a_{mn} \end{bmatrix} =$$

$$= \underbrace{u_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + u_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{}$$
(ker smo v polju, je * komutativno).

↳ linearna kombinacija stolpcev matrike A.

$\text{Im } A$ je torej linearna ogrinjača stolpcev matrike A.
$\text{Im } A$ = stolpčni prostor matrike $A =: \underbrace{\text{Col}}_{} A$ (def. za $\text{Col } A$)
↳ column

//

↳ $r(A) = \dim \operatorname{Im} A$ = največje število linearno neodvisnih stolpcev matrike A.

Trditev: linearna preslikava L je injektivna ↳ $Lu_1 = Lu_2 \Rightarrow u_1 = u_2$

$\Longleftrightarrow$

$\operatorname{Ker} L = \{0\}$

Dokaz

($\Longrightarrow$) predp. L injektivna

$\operatorname{Ker} L = \{0\}$ vzemimo poljuben $u \in \operatorname{Ker} L \Rightarrow \underbrace{Lu = 0 = L0}_{\text{isti el. zaradi inj.}} \Rightarrow u = 0$

($\Longleftarrow$) predp. $\operatorname{Ker} L = \{0\}$

L injektivna $Lu_1 = Lu_2 \overset{L.P.}{\Rightarrow} L(u_1 - u_2) = 0 \Rightarrow u_1 - u_2 \in \operatorname{Ker} L$

$\overset{\operatorname{Ker} L = \{0\}}{\Longrightarrow} u_1 - u_2 = 0 \Rightarrow u_1 = u_2$

[OSNOVNA FORMULA]

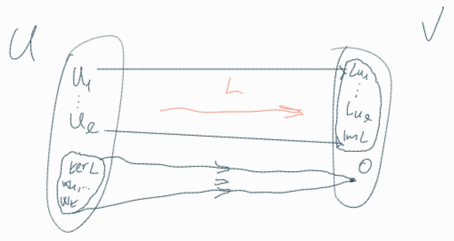
Izrek: let $L: U \to V$ L.P.

tedaj je $\dim \operatorname{Ker} L + \dim \operatorname{Im} L = \dim U$

$$n(L) + r(L) = \dim U$$

(pri matrikah $n(A) + r(A) = \dim F^n = n$ za $m \times n$ matriko A)

Dokaz:



Baza jedra: $w_1, \dots w_k$

let $u_1, \dots u_l$ dopolnitev $w_1, \dots w_k$ do baze U

$\dim U = k + l$

$k \to \dim \operatorname{Ker} A = n(A)$

$l \to$ dokazati je treba še, da je $l = \dim \operatorname{Im} A = r(A)$

Konstruirali bomo bazo za $\operatorname{Im} L$, ki ima l elementov.

dokazati je treba, da so $Lu_1, \dots, Lu_l$ baza za $\operatorname{Im} L$.

Zakaj so ogrodje? vzemimo poljuben $v \in \operatorname{Im} L$.

$\overset{\operatorname{Im} L}{\Longrightarrow} v = Lu$ za $u \in U$

razvijmo u po bazi za L:

$$u = \alpha_1 w_1 + \dots + \alpha_k w_k + \beta_1 u_1 + \dots + \beta_l u_l$$

apply L to both sides of the equation

$$v = Lu = L(\alpha_1 w_1 + \dots + \alpha_k w_k + \beta_1 u_1 + \dots + \beta_l u_l)$$

$\overset{L.P.}{=} \alpha_1 Lw_1 + \dots + \alpha_k Lw_k + \beta_1 Lu_1 + \dots + \beta_l Lu_l =$

$= \beta_1 Lu_1 + \dots + \beta_l Lu_l$ zato je $\vec{u}$ ves ogrodje.

↳ $\operatorname{Lin}\{Lu_1, \dots, Lu_l\} = \operatorname{Im} L$

Zakaj pa je $Lu_1, \dots, Lu_\ell$ linearno neodvisna?

$$\gamma_1 Lu_1 + \dots + \gamma_\ell Lu_\ell = 0 \overset{?}{\Longrightarrow} \gamma_1 = \dots = \gamma_\ell = 0$$

$$L(\gamma_1 u_1 + \dots + \gamma_\ell u_\ell) = \gamma_1 Lu_1 + \dots + \gamma_\ell Lu_\ell = 0$$

$$\Rightarrow \gamma_1 u_1 + \dots + \gamma_\ell u_\ell \in \ker L$$

lahko razvijemo po bazi jedra:

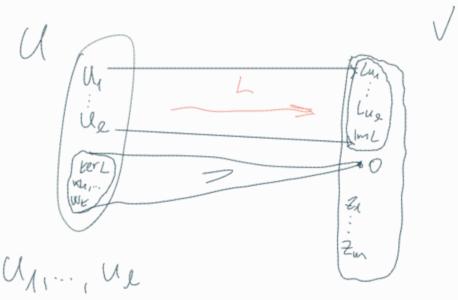
$$\gamma_1 u_1 + \dots + \gamma_\ell u_\ell = \delta_1 w_1 + \dots + \delta_k w_k \quad \text{za neke } \vec{\delta}.$$

$$\gamma_1 u_1 + \dots + \gamma_\ell u_\ell - \delta_1 w_1 - \dots - \delta_k w_k = 0$$

$$\xRightarrow{\vec{u}, \vec{w} \text{ L.N.}} \gamma_1 = \dots = \gamma_\ell = 0 = -\delta_1 = \dots = -\delta_k = 0$$

torej $\gamma_1 Lu_1, \dots, \gamma_\ell Lu_\ell$ so L.N.

vrnimo se k sliki:



$z_1, \dots, z_m$ je dopolnitev $Lu_1, \dots, Lu_\ell$ do baze V.

$B = w_1, \dots, w_k, u_1, \dots, u_\ell$ baza za U

$C = Lu_1, \dots, Lu_\ell, z_1, \dots, z_m$ je baza za V.

$$[L]_{C \leftarrow B} = ?$$

$$\begin{cases} Lu_1 = 1 \cdot Lu_1 + \dots + 0\,Lu_\ell + 0\,z_1 + \dots + 0\,z_m \\ \vdots \\ Lu_\ell = 0\,Lu_1 + \dots + 1\,Lu_\ell + 0\,z_1 + \dots + 0\,z_m \\ Lw_1 = 0\,Lu_1 + \dots + 0\,Lu_\ell + 0\,z_1 + \dots + 0\,z_m \\ \vdots \\ Lw_k = 0\,Lu_1 + \dots + 0\,Lu_\ell + 0\,z_1 + \dots + 0\,z_m \end{cases}$$

$$= \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & & & \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & & 0 \end{bmatrix} = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix} \qquad r(L) = \ell$$

S tako primerno izbiro baz U in V je matrika preslikave precej preprosta.

Kaj pa, če je L matrika? Naj bo A.

$A \in M_{m,n}(F)$.

stolpci baza U

$$P = [u_1 \ \dots \ u_\ell \ w_1 \ \dots \ w_k]$$ obrnljiva

stolpci baza V

$$Q = [Au_1 \ \dots \ Au_\ell \ z_1 \ \dots \ z_p]$$ obrnljiva

pišimo p, m je že višina matrike

$$AP = ? = [Au_1 \ \dots \ Au_\ell \ \underbrace{Aw_1}_{= 0} \ \dots \ \underbrace{Aw_k}_{= 0}]$$

so v jedru A.

$$Q \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix} = [Au_1 \ \dots \ Au_\ell \ z_1 \ \dots \ z_p] \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix} = [Au_1 \ \dots \ Au_\ell \ 0 \ \dots \ 0]$$

$$\Rightarrow AP = Q \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}.$$

$$Q^{-1}AP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

[EKVIVALENTNOST MATRIK]

matriki A in B sta ekvivalentni (oznaka $A \sim B$) natanko tedaj, ko $\exists$ obrnljivi P, Q, $\ni$: $B = PAQ$

Primer: Dokazali smo, da je vsaka matrika A ekvivalentna matriki $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, kjer $r =$ rang od A. (*)

Dokažimo ekvivalenčnost te relacije $\sim$.

- refleksivnost: $A \sim A = 1$: $A = I_m \cdot A \cdot I_n$ za $A \in M_{m,n}(F)$.
- simetričnost: $A \sim B \overset{?}{\Rightarrow} B \sim A$

$\Downarrow$

$\exists$ obr. P, Q, $\ni$: $B = PAQ$

$\Rightarrow A = P^{-1} B Q^{-1} \Rightarrow B \sim A$

- tranzitivnost $A \sim B \wedge B \sim C \Rightarrow A \sim C$

$\Downarrow$ $\Downarrow$

$B = PAQ$ $C = SBT$

$C = (SP)A(QT)$

produkt obrnljivih matrik je obrnljiva matrika

$\square$ ekvivalenčnost

Izrek: Dve matriki sta ekvivalentni, ko imata enako velikost (m, n) in enak rang.

$(\Leftarrow)$: predp.: A in B imata enako velikost (m, n) in enak rang r.

Primer (*) nam pove, da

$$A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \overset{\text{ekvivel}}{\Rightarrow} A \sim B$$

$(\Rightarrow)$: predp. matriki sta ekvivalentni: imata enak rang in velikost

po def. $\exists$ obr. P, Q $\ni$: $B = PAQ$

let (m, n) je velikost A. $\Rightarrow P$ je $m \times m$ matrika
(m vrstic, n stolpcev) Q je $n \times n$ matrika

$\Rightarrow B$ je $m \times n$ matrika zaradi delovanja matričnega množenja

imata enako velikost.
kaj pa rang?

$$r(B) = \boxed{r(\underbrace{PA}_{C}Q) \overset{?}{=} r(\underbrace{PA}_{C})} \overset{?}{=} r(A)$$

↳ zadošča dokazati, da je $\mathrm{Im}(CQ) = \mathrm{Im}(C)$

$\mathrm{Im}(CQ) \stackrel{?}{=} \mathrm{Im}(C)$

$u \in \mathrm{Im}(CQ) \iff u = (CQ)v$ za nek v

$\iff u = Cv'$ za nek v'

$\stackrel{Q \text{ obrnljiva}}{\iff} u \in \mathrm{Im}\, C$

torej $\mathrm{Im}(CQ) = \mathrm{Im}(C)$

torej $r(PAQ) = r(PA)$

zato je $r(PA) = r(A)$.

zadošča dokazati da je $\ker(PA) = \ker A$

nato lahko uporabimo

$\dim \mathrm{Im}\, A + \dim \ker A = \dim F^n = n$

$\dim \mathrm{Im}\, PA + \dim \ker PA = \dim F^n = n$

iz enakosti izrazov $\implies \dim \mathrm{Im}\, PA = \dim \mathrm{Im}\, A$

$u \in \ker PA \iff PAu = 0 \stackrel{P \text{ obr.}}{\iff} Au = 0 \iff u \in \ker A \quad \forall u.$

torej je res $\ker PA = \ker A$

torej je $r(PA) = r(A)$, s čimer dokažemo izrek.

za naslednjič: $r(A) = r(A^T)$.