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[4.4 Binomska vrsta in Catalanova števila]

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \frac{1}{(1+x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

Kaj pa $(1+x)^{1/2}$ oz. $\sqrt{1+x}$? Želimo $\sqrt{1+x}^{\,2} = 1+x$

in prosti koeficient naj bo 1.

Def.:

Posplošeni binomski koeficient:

$\lambda \in K$, $n \in \mathbb{N}$

komut. obseg $\cong$ polje s char 0

$$\binom{\lambda}{n} := \frac{\lambda^{\underline{n}}}{n!} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}$$

$n! \neq 0$, ker char $= 0$

Primeri:

$$\binom{-k}{n} = \frac{(-k)(-k-1)\cdots(-k-n-1)}{n!}\cdot\frac{(k-1)!}{(k-1)!} =$$

$$= (-1)^n \frac{(n+k-1)!}{n!\,(k-1)!} = (-1)^n \binom{n+k-1}{k-1}$$

$$\binom{\lambda}{0} = \frac{1}{1!} = 1 \qquad \binom{\lambda}{1} = \lambda \qquad \binom{\lambda}{2} = \frac{\lambda(\lambda-1)}{2}$$

$$\binom{-1}{n} = (-1)^n$$

za $n \geq 1$:
$$\binom{1/2}{n} = \frac{(1/2)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} =$$

$$(2n)!! = 2\cdot4\cdot6\cdots(2n) = 2^n \cdot n!$$
$$(2n+1)!! = 1\cdot3\cdot5\cdots(2n-1)$$

$$= \frac{(-1)^{n-1}(2n-3)!!}{2^n\, n!}\cdot\frac{(2n-2)!!}{(2n-2)!!} =$$

izpostavi 2^{2n-1}

$$= \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}\, n!\,(n-1)!} = \frac{(-1)^{n-1}}{2^{2n-1}\, n}\binom{2n-2}{n-1}$$

Def.: Binomska vrsta

$$B_\lambda(x) := \sum_{n=0}^{\infty} \binom{\lambda}{n} x^n \qquad \text{za } \lambda \in K$$

Primeri:

$$B_k(x) = (1+x)^k \quad \text{za } k \in \mathbb{N}$$

$$B_{-k}(x) = \sum_{n=0}^{\infty}\binom{-k}{n}x^n = \sum_{n=0}^{\infty}(-1)^n\binom{n+k-1}{k-1}x^n = (1+x)^{-k}$$

Radi bi pisali: $B_\lambda(x) = (1+x)^\lambda$. Pričakujemo $(1+x)^\lambda(1+x)^\mu = (1+x)^{\lambda+\mu}$.

Lema: $(a+b)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} a^{\underline{n-k}} b^{\underline{k}}$ za $n \in \mathbb{N}$, $a, b \in K$.

dokaz z indukcijo

$n=0$: $(a+b)^{\underline{0}} = 1 = 1$ ✓

$n-1 \to n$: $(a+b)^{\underline{n}} = (a+b)^{\underline{n-1}} \cdot a+b-n+1 \overset{IP}{=}$

$= \left(\sum_k \binom{n-1}{k} a^{\underline{n-1-k}} b^{\underline{k}}\right)(a+b-n+1) =$

$a(a-1)\cdots(a-n+1+k+1)$

$= \left(\sum_k \binom{n-1}{k} a^{\underline{n-1-k}} b^{\underline{k}}\right)\left((a-n+k+1) + (b-k)\right) =$

$b(b-1)\cdots(b-b+1)$

$= \sum_k \binom{n-1}{k} a^{\underline{n-k}} b^{\underline{k}} + \underbrace{\sum_k \binom{n-1}{k} a^{\underline{n-1-k}} b^{\underline{k+1}}}_{\parallel\, k \to k-1} =$

$\sum_k \binom{n-1}{k-1} a^{\underline{n-k}} b^{\underline{k}}$

$= \sum_k \binom{n}{k} a^{\underline{n-k}} b^{\underline{k}}$

Izrek: (1) $B_\lambda(x) B_\mu(x) = B_{\lambda+\mu}(x)$ $\quad \lambda, \mu \in K$

(2) $(1+x) B_\lambda'(x) = \lambda B_\lambda(x)$ $\quad \lambda \in K$

Dokaz: (1) $[x^n] B_\lambda(x) B_\mu(x) = \sum_{k=0}^{n} \binom{\lambda}{n-k}\binom{\lambda}{k} = \sum_{k=0}^{n} \frac{\lambda^{\underline{n-k}}}{(n-k)!} \cdot \frac{\mu^{\underline{k}}}{k!} =$

$n-k$-ti koef

konvolucija

k-ti koef

$= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^{\underline{n-k}} \mu^{\underline{k}} \overset{\text{lema}}{=} \frac{(\lambda+\mu)^{\underline{n}}}{n!} = \binom{\lambda+\mu}{n} - B_{\lambda+\mu}(x)$

(2) $B_\lambda(x) = \sum_n \binom{\lambda}{n} x^n$

$B_\lambda'(x) = \sum_n (n+1)\binom{\lambda}{n+1} x^n$ $\quad n \to n-1$

$(1+x) B_\lambda'(x) = \sum_n (n+1)\binom{\lambda}{n+1} x^n + \sum_n n\binom{\lambda}{n} x^n =$

$= \sum_n \left(\frac{\lambda^{\underline{n+1}}}{n!} + \frac{n\lambda^{\underline{n}}}{n!}\right) x^n = \lambda B_\lambda(x)$

$\lambda^{\underline{n+1}} + n\lambda^{\underline{n}} = \lambda^{\underline{n}}(\lambda \cancel{-n} \cancel{+n}) = \lambda \lambda^{\underline{n}}$

Zato vpeljemo novo oznako $(1+x)^\lambda = B_\lambda(x)$

$$(1+x)^\lambda \cdot (1+x)^\mu = (1+x)^{\lambda+\mu}$$

$$\left((1+x)^\lambda\right)' = \lambda(1+x)^{\lambda-1}$$

$$\left((1+x)^{\frac{1}{2}}\right)^2 = 1+x = \sqrt{1+x}^{\,2} = \sqrt{1+x}\sqrt{1+x}$$

C_n (catalanovo število) je število pravilnih postavitev oklepajev na nizu $t_0\, t_1 \dots t_n$: oklepaj zajame par!

$n=2$: $(t_0\ t_1)\ t_2 \qquad t_0\ (t_1\ t_2)$

$n=3$:

$((t_0\ t_1)\ t_2)\ t_3$

$(t_0\ t_1)(t_2\ t_3)$

$(t_0\ (t_1\ t_2))\ t_3$

$t_0\ ((t_1\ t_2)\ t_3)$

$t_0\ (t_1\ (t_2\ t_3))$

$C_0 = 1$

$C_1 = 1$

$C_2 = 2$

$C_3 = 5$

$C_4 = 14$

$C_5 = 42$

$C_6 = 132$

$\dots$

$$C_{n+1} = \overbrace{\sum_{k=0}^{n}}^{\text{izbira } \boxed{\text{zadnje operacije}}} C_k \cdot C_{n-k} \quad \text{velja od } n \geq 0$$

$$(t_0\ t_1 \quad)(\quad t_{n+1})$$

$\boxed{\text{zadnja operacija}}$

$$C_3 = C_0 \cdot C_2 + C_1 \cdot C_1 + C_2 \cdot C_0 = 1\cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$$

$$F(x) = \sum_n C_n x^n$$

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad / \cdot x^{n+1} \quad / \sum_{n=0}^{\infty}$$

$$\sum_{n=0}^{\infty} C_{n+1} x^{n+1} = F(x) - C_0 = F(x) - 1 = x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^n = x F(x)^2$$

$$F(x) = 1 + x F^2(x)$$

oz. $\quad x F^2(x) - F(x) + 1 = 0$

$$F(x) \stackrel{?}{=} \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n-1} n} \binom{2n-2}{n-1} (-1)^n 2^{2n} x^n =$$

$$= 1 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n$$

$$\frac{1+\sqrt{1-4x}}{2x} = \frac{1}{x} + \ldots \notin K[[x]]$$

$$\frac{1-\sqrt{1-4x}}{2x} = \frac{\cancel{1}-\cancel{1}+\cancel{2}\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^n}{\cancel{2}x} = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^{n-1} =$$

$$n \longrightarrow n-1$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n$$

Izrek: formula za n-to Catalanovo število:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Opomba: zakaj obrazec za ničle kv. en. deluje?

$$\left(2xF(x)-1-\sqrt{1-4x}\right)\left(2xF(x)-1+\sqrt{1-4x}\right) =$$

$$= \left(2xF(x)-1\right)^2 - \left(\sqrt{1-4x}\right)^2 = 4x^2F^2(x) - 4xF(x) + \cancel{1} - \cancel{1} + 4x =$$

$$= 4x\left(xF^2(x) - F(x) + 1\right) = 0$$

ker $K[[x]]$ nima deliteljev niča, mora biti ena od $2xF(x)-1-\sqrt{1-4x}$ in $2xF(x)-1+\sqrt{1-4x}$ enaka 0.

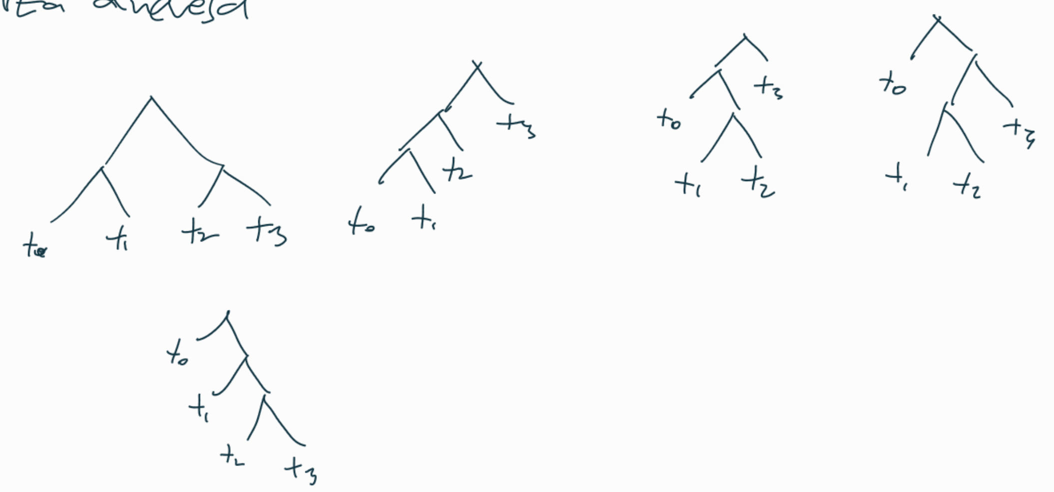
Ker je konst. koef. od $2xF(x)-1-\sqrt{1-4x}$ enak -2, ta ni ničelna. Zato mora biti $2xF(x)-1+\sqrt{1-4x}$ enaka 0.

$$C_3 = \frac{1}{4}\binom{6}{3} = \frac{6\cdot5\cdot4}{4\cdot6} = 5$$

$$C_4 = \frac{1}{5}\binom{8}{4} = \frac{\overset{2}{\cancel{8}}\cdot7\cdot\cancel{6}\cdot\cancel{8}}{\cancel{4!}\cdot\cancel{8}} = 2\cdot7 = 14$$

Opomba: Catalanova števila predstavljajo veliko različnih struktur. večino dvojiška drevesa

$C_3 = 5$



Dyckove poti:

gor/dol/vedno se vrnemo na x os in nikoli ne gremo pod x os

RPN!

triangulacija $(n+2)$-kotnika:

rekurzija:

Stanley: Enumerative Combinatorics II
opiše 66 struktur za Catalanova števila,
na www pa še >100 struktur.

[4.5 Rodovne funkcije razčlenitev]

$p(n)$ ... št. razčl. št. n

$p_k(n)$ ... št. razčl. št. n s k členi

$\overline{p}_k(n)$ ... št. r. št. n z največ k členi

$$p_k(n) = p_{k-1}(n-1) + p_k(n-1)$$

$$\overline{p}_k(n) = \overline{p}_{k-1}(n) + \overline{p}_k(n-k)$$

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) \ldots$$

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3} = \underline{(1+x+x^2+\dots)}\,\underline{(1+x^2+x^4+\dots)}\cdot\underline{(1+x^3+x^6+\dots)}$$

koef. pri x^5:

$$\begin{array}{ccc} \underline{x^0} & \underline{x^2} & \underline{x^3} \\ x^1 & x^4 & x^0 \\ x^2 & x^0 & x^3 \\ x^3 & x^2 & x^0 \\ x^5 & x^0 & x^0 \end{array} \Bigg\} \; 5 \leftarrow \begin{array}{l} 32 \\ 221 \\ 311 \\ 2111 \\ 11111 \end{array}$$

razčlenitve 5 s členi $\in [1,3]$.

splošen primer

$$[x^n]\,\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\ \dots\ \cdot\frac{1}{1-x^k}$$

$$x^{m_1\cdot 1}\quad x^{m_2\cdot 2}\quad \dots\quad x^{m_k\cdot k}$$

$$\underbrace{m_1\cdot 1 + m_2\cdot 2 + \dots + m_k\cdot k = n}$$

število razčlenitev n s členi $\in [1,k]$

diagrami kateri niso širši od k.

če transponiramo, dobimo diagrame z največ k vrsticami; razčlenitve n z največ k členi.

št. razčlenitev n z do k členi $= \bar{p}_k(n)$

kaj je rodovna f-ja za $\bar{p}_k(n)$?

Izrek: $$\sum_n \bar{p}_k(n)\,x^n = \frac{1}{\prod_{i=1}^{k}(1-x^i)}$$

kaj pa $\sum_n p_k(n)\,x^n$?

ni enic (x^0)

to je pa $$\frac{x}{1-x}\cdot\frac{x}{1-x^2}\cdot\frac{x}{1-x^3}\ \cdots\ \frac{x}{1-x^k} = (x+x^2+\dots)(x+x^2+x^4)\dots =$$

$$= \frac{x^k}{\prod_{i=1}^{k}(1-x^i)}$$

Kaj pa $p(n)$:

$$\sum_n p(n)\, x^n = \frac{1}{\prod_{i=1}^{\infty}(1-x^i)}$$

za koeficient pri x^n potrebujemo še vedno le končne vsote; t.j. i od 1 do n.

$$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\cdots$$

$$[x^5]: \quad \begin{array}{ccc} x^5 & x^0 & x^0 \\ x^3 & x^2 & x^0 \\ & \vdots & \end{array}$$

$$x^0\, x^0\, x^0\, x^0\, x^5\, x^0\, x^0\, x^0$$

Let $o(n)$ število razčlenitev n s samimi lihimi členi:

$$\sum_n o(n)\, x^n = \frac{1}{\prod_{i \text{ lih}}(1-x^i)}$$

n	$o(n)$
1	1
2	1
3	2
4	2

$d(n)$ št. razčl. s samimi različnimi členi

distinct

n	$d(n)$
1	1
2	1
3	2
4	2

$$\sum_n d(n)\, x^n = (1+x)(1+x^2)(1+x^3)\cdots = \frac{\prod_i (1+x^i)(1-x^i)}{\prod_i (1-x^i)} =$$

$$= \frac{\prod_i (1-x^{2i})}{\prod_i (1-x^i)} = \frac{1}{\prod_{i \text{ lih}}(1-x^i)} = o(n)$$

$$\Longrightarrow \quad d(n) = o(n)$$