KOMBPFMF 2024-11-18

meta: 3.1. 2. kol.
13.1. zadnje predav.
22.1. 1. izpit
5.2. 2. izpit
27.1. ustni za kolokvije
28.1.–14.2. prof. odsoten
17.2.–9.9. ustni zopet

[4.3 UPORABA RODOVNIH fj pri reševanju rekurzivnih enačb]

$F_n = F_{n-1} + F_{n-2}$ $F_0 = F_1 = 1$ fibbonacijevo

$$F(x) = 1, 1, 2, 3, 5, 8, 13, \dots$$
$$x \cdot F(x) = 0, 1, 1, 2, 3, 5, 8, 13, \dots$$
$$x^2 F(x) = 0, 0, 1, 1, 2, 3, 5, 8, 13, \dots$$

$$F(x) = 1 + x F(x) + x^2 F(x)$$
$$F(x)(1 - x - x^2) = 1$$
$$F(x) = \frac{1}{1 - x - x^2}$$

Primeri:

$\boxed{\text{NE BERITE MI MISLI! LP Anton}}$

(1.) $a_n = 2a_{n-1}$; $a_0 = 1$, $n \geq 1$

$$a_n = 2a_{n-1} \;/ x^n \;/ \sum_{n=1}^{\infty}$$ in izrazimo z $F(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

$$F(x) - \overset{1}{a_0} = 2 \sum_{n=1}^{\infty} a_{n-1} x^n = 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 2x F(x)$$

$$F(x) - 1 = 2x F(x)$$
$$F(x)(1 - 2x) = 1$$
$$F(x) = \underbrace{\frac{1}{1-2x}}_{\text{geom. vrsta}} = \sum_{n=0}^{\infty} 2^n x^n \Rightarrow a_n = 2^n$$

(2.) $a_n = 3a_{n-1} - 2a_{n-2}$ $a_0 = 1$ $a_1 = 4$, $n \geq 2$

$$/ \cdot x^n \;/ \sum_{n=2}^{\infty}$$ $F(x) = \sum_{n=0}^{\infty} a_n x^n$

1, 4, 10, 22, 46

$$F(x) - 1 - 4x = 3x(F(x) - 1) - 2x^2 F(x)$$

tako bi storili pri ANA

$$F(x) = \frac{1 + 4x - 3x}{1 - 3x + 2x^2} = \frac{1+x}{1 - 3x + 2x^2} = \boxed{\frac{1+x}{2(x - 1/2)(x-1)} = \frac{A}{x - 1/2} + \frac{B}{x-1}}$$

pri KOMB po rabi:

$$\frac{1+x}{1-3x+2x^2} = \frac{}{(1-x)(1-2x)}$$

pri analizi:

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

pri kombinatoriki:

$$c+bx+ax^2 = c(1-y_1x)(1-y_2x)$$

$$y_{1,2} = \frac{1}{x_{1,2}} = \frac{2a}{-b \pm \sqrt{b^2-4ac}} \cdot \frac{-b \mp \sqrt{D}}{-b \mp \sqrt{D}} =$$

$$y^3-3y+2 = (y-1)(y-2)$$

$$\Downarrow$$

$$(1-x)(1-2x)$$

$$= \frac{\cancel{2a}(-b \mp \sqrt{b^2-4ac})}{\cancel{b^2-b^2+4ac}_{\;2}} = \frac{-b \mp \sqrt{b^2-4ac}}{2}$$

$$\|$$

$y_{1,2}$ sta rešitvi obrnjene kvadratne

tj. cy^2+by+a

$$\ldots = \frac{1+x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$ najdimo A, B.

1. način: seštejemo desno stran in preuredimo

$$\frac{1+x}{\cancel{(1-x)(1-2x)}} = \frac{A(1-2x)+B(1-x)}{(1-x)(1-2x)} = \frac{A+B+x(-2A-B)}{\cancel{(1-x)(1-2x)}}$$

$$1 = A+B \qquad A=-2;\ B=3$$
$$1 = -2A-B$$

2. način: metoda pokrivanja (ne dela vedno)

- obe strani množimo z $(1-x)$ in vstavimo $x=1$

desna stran: ostane A

leva: $2/_{-1} = -2 \implies A=-2$

- obe strani množimo z $(1-2x)$ in vstavimo $x=1/2$

desno ostane B

levo: $\frac{3/2}{1/2} = 3$

ne dela pri ponovljenih ničlah

$$F(x) = -\frac{2}{1-x} + \frac{3}{1-2x} = -2\sum_{n=0}^{\infty} x^n + 3\sum_{n=0}^{\infty} 2^n x^n$$

$$a_n = 3 \cdot 2^n - 2$$

NTSB1RZ100

(3) $F_n = F_{n-1} + F_{n-2} \quad n \geq 2 \quad F_0 = F_1 = 1$

$$/ \cdot x^n \; / \sum_{n=2}$$

$$F(x) - 1 - x = x\,(F(x) - 1) + x^2 F(x)$$

$$F(x) = \frac{1 + \cancel{x - x}}{1 - x - x^2} = \frac{1}{1 - x - x^2} = \frac{1}{(1 - y_+ x)(1 - y_- x)}$$

$y_{+,-}$ sta ničli $y^2 - y - 1$

$$y_{+,-} = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{\frac{1}{1 - \frac{y_-}{y_+}} \; / \cdot y_+}{1 - y_+ x} + \frac{\frac{1}{1 - \frac{y_+}{y_-}} \; / \cdot y_-}{1 - y_- x}$$

$$\frac{1}{y_+ - y_-}\left(\frac{y_+}{1 - y_+ x} - \frac{y_-}{1 - y_- x}\right)$$

$$y_+ - y_- = \sqrt{5}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

(4) $a_n = -2a_{n-1} - 2a_{n-2} \quad n \geq 2 \qquad a_0 = 1 \,;\; a_1 = -1$

$$/ \cdot x^n \; / \sum_{n=2}$$

$$F(x) - 1 + x = -2x\left(F(x) - 1\right) - x^2 F(x)$$

$$F(x) = \frac{1 - x + 2x = 1 + x}{1 + 2x + 2x^2}$$

$y + 2y + 2$

$$y_{1,2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$y_{1,2} = -1 \pm i$$

$$\frac{1+x}{(1 - (-1+i)x)(1 - (-1-i)x)}$$

$$= \frac{\frac{1 + \frac{1}{-1+i}}{1 - (-1-i)\frac{1}{-1+i}}}{1 - (-1+i)x} + \frac{\frac{1 + \frac{1}{-1-i}}{1 - (-1+i)\frac{1}{-1-i}}}{1 - (-1-i)x} =$$

$$= \frac{\frac{\cancel{i}}{2\cancel{i}}}{1-(-1+i)x} + \frac{\frac{\cancel{-i}}{\cancel{-2i}}}{1-(-1-i)x} =$$

$$= \frac{1}{2}\left(\frac{1}{1-(-1+i)x} + \frac{1}{1-(-1-i)x} \right)$$

$$z = x+iy = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$$

$$z^n = (x+iy)^n = r^n e^{in\varphi} = r^n(\cos n\varphi + i\sin n\varphi)$$

(de Moivre)

NTSBPKDRS

$$F_n = \frac{1}{2}\left((-1+i)^n + (-1-i)^n\right) =$$

$$(-1+i)^n = \left(\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right)^n =$$

$$= 2^{n/2}\left(\cos\frac{3\pi n}{4} + i\sin\frac{3\pi n}{4}\right)$$

$$(-1-i)^n = 2^{n/2}\left(\cos\frac{3\pi n}{4} - i\sin\frac{3\pi n}{4}\right)$$

$$a_n = 2^{\frac{n}{2}}\cos\frac{3\pi n}{4}$$

$1, -1, 0, 2, -4, 4, 0, -8, 16$

(5) $a_n = 6a_{n-1} - 9a_{n-2} \quad n \geq 2 \quad a_0 = 1 \quad a_1 = 5$ NTSBPKDRS NOSUHO

$$F(x) - 1 - 5x = 6x(F(x)-1) - 9x^2 F(x)$$

$$F(x) = \frac{1-x}{1-6x+9x^2} \qquad y^2-6y+9 = (y-3)^2$$

$$\frac{1-x}{(1-3x)^2} = \frac{A}{1-3x} + \frac{B}{(1-3x)^2} =$$

za B množi z $(1-3x)^2$ in vstavi v x

$B = 2/3$

$$= \frac{A(1-3x) + \frac{2}{3}}{(1-3x)^2}$$

$$-3A = -1$$

$$A = \frac{1}{3}$$

$$F(x) = \frac{1/3}{1-3x} + \frac{2/3}{(1-3x)^2} =$$

$$\frac{1}{(1-3x)^2} = ?$$

1. način: konvolucijsko množenje 3^n

$$\frac{1}{1-3x} \cdot \frac{1}{1-3x} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 3^k \cdot 3^{n-k}\right)x^n =$$

$$= \sum_{n=0}^{\infty}(n+1)3^n x^n$$

2. način: odvajanje

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n \quad /'$$

$$\frac{3}{(1-3x)^2} = \sum_{n=0}^{\infty} (n+1)\, 3^{n+1} x^n$$

$$\frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} (n+1)^3 x^n$$

3. način: preko šibkih kompozicij

$$\frac{1}{(1-x)^k} = \underbrace{\frac{1}{1-x} \cdot \ldots \cdot \frac{1}{1-x}}_{k\text{-krat}} =$$

$$= \overbrace{(1+x+x^2+\dots)(1+x+x^2+\dots)\cdots(1+x+x^2+\dots)}.$$

koef. pri x^n = ? primer: $k=3$, $n=5$

šibke kompozicije dolžine 3 velikosti 5

$$\begin{cases} x^3 \cdot x^2 \cdot x^0 = x^5 \\ x^3 \cdot x^1 \cdot x^1 = x^5 \\ x^1 \cdot x^0 \cdot x^4 = x^5 \\ \ldots \end{cases}$$

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

↳ št. šibkih kompozicij s k členi

$k=3$: $$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$$

D.N. dokaži z indukcijo

vrnimo se k nalogi:

$$F(x) = \frac{1/3}{1-3x} + \frac{2/3}{(1-3x)^2}$$

$$a_n = \frac{1}{3} \cdot 3^n + \frac{2}{3}(n+1) \cdot 3^n = \frac{3^n(2n+3)}{3}$$

V resnici rešujemo homogene linearne rekurzivne enačbe s konstantnimi koeficienti:

$$c_d a_n + c_{d-1} a_{n-1} + \dots + c_0 a_{n-d} = 0$$

$$\forall i \in \{0..d\}: c_i \in \mathbb{C} \quad ; \quad c_d, c_0 \neq 0$$

Red te HLRESKK je d.

$$F(x) = \sum_n a_n x^n \qquad / \cdot x^n / \sum_{n=d}^{\infty}$$

$$c_d \Big(F(x) - \underbrace{a_0 - a_1 x - \dots - a_{d-1} x^{d-1}}_{\text{stopnja } < d}\Big) +$$

$$+ c_{d-1} x \Big(F(x) \underbrace{- a_0 - a_1 x - \dots - a_{d-2} x^{d-2}}_{\text{stopnja } < d-1}\Big) +$$

$$+ c_{d-2} x^2 \Big(F(x) \underbrace{- a_0 - a_1 x - \dots - a_{d-3} x^{d-3}}_{< d-2}\Big) + \dots$$

$$\dots$$

$$+ c_1 x^{d-1} (F(x) - a_0) + c_0 x^d F(x) = 0$$

$$\Rightarrow \quad F(x) = \frac{\overset{\text{stopnja } < d}{p(x)}}{\underbrace{c_d + c_{d-1} x + \dots + c_1 x^{d-1} + c_0 x^d}_{\text{stopnja } = d, \text{ ker } c_0 \neq 0}}$$

$$c_d + c_{d-1} x + \dots + c_1 x^{d-1} + c_0 x^d = c_d (1 - \lambda_1 x)^{\alpha_1} \cdot \dots \cdot (1 - \lambda_k x)^{\alpha_k}$$

$\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}$ so ničle $c_d + c_{d-1} x + \dots + c_0 x^d$

$$c_d + c_{d-1} \frac{1}{\lambda_i} + \dots + c_0 \frac{1}{\lambda_i^d} = 0 \qquad / \cdot \lambda_i^d$$

$$c_d \lambda_i^d + c_{d-1} \lambda_i^{d-1} + \dots + c_0 = 0$$

Torej $\lambda_1, \dots, \lambda_k$ so ničle karakterističnega polinoma

$$c_d \lambda^d + c_{d-1} \lambda^{d-1} + \dots + c_0$$

$$F(x)=\frac{p(x)}{C_d\prod_{i=1}^{t}(1-\lambda_i x)^{\alpha_i}}=\sum_{i=1}^{t}\sum_{j=1}^{\alpha_i}\frac{A_{i,j}}{(1-\lambda_i x)^j}=\sum_{i=1}^{t}\sum_{j=1}^{\alpha_i}A_{ij}\sum_{n=0}^{\infty}\binom{n+j-1}{j-1}\lambda_i^n x^n$$

$$\Rightarrow a_n=\sum_{i=1}^{t}\left(\underbrace{\sum_{j=1}^{\alpha_i}A_{ij}\underbrace{\binom{n+j-1}{j-1}}_{\text{polinom v } n \text{ stopnje } j-1}}_{\text{polinom stopnje} < \alpha_i \text{ v } n}\lambda_i^n\right)$$

Izrek: rešitev HLREKE je $a_n=\sum_{i=1}^{t}p_i(n)\lambda_i^n$,

kjer so $\lambda_1,\dots,\lambda_t$ ničle karakterističnega polinoma večkratnosti $\alpha_1,\dots,\alpha_t$; p_i pa je polinom stopnje $<\alpha_i$

(dokazan zgoraj).

$\Rightarrow$ to služi kot nastavek za linearne rekurzije.

Primeri:

(1) $a_n=2a_{n-1}$; $n\geq 1$; $a_0=1$

$a_n-2a_{n-1}=0$

kar. pol.: $\lambda-2$ ničla 2 večkratnosti 1

$\to$ polinom stopnje <1

$a_n=A\cdot 2^n$

$a_0=1=A\cdot 2^0=A$

$A=1 \to a_n=2^n$

(2) $a_n-3a_{n-1}+2a_{n-2}=0$ $n\geq 2$ $a_0=1$ $a_1=4$

$\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2)$ ničli 1,2 večkratnosti 1

$a_n=A\,1^n+B\,2^n$

$a_0=A+B=1$ $a_1=A+2B=4$ $B=3$ $A=-2$

$\Rightarrow a_n=3\cdot 2^n-2$

(3) $F_n - F_{n-1} - F_{n-2} = 0 \qquad n \geq 2 \quad F_0 = F_1 = 1$

$$\lambda^2 - \lambda - 1 = 0 \qquad \text{ničli: } \frac{1 \pm \sqrt{5}}{2} \quad \text{večkr. } 1$$

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \ldots$$

(5) $a_n - 6a_{n-1} + 9\,a_{n-2} = 0 \qquad a_0 = 1 \quad a_1 = 5$

$$\lambda^2 - 6 + 9 = (\lambda - 3)^2$$

$$a_n = (A + Bn)3^n \quad \ldots$$

Reševanje nekaterih nehomogenih linearnih rekurzivnih enačb s konstantnimi koeficienti.

NE MI BRAT MISLI!

$$c_d a_n + \cdots + c_0 a_{n-d} = r(n) \cdot \boxed{\lambda}^n$$

Rešitev je vsota rešitve homogenega dela in partikularne rešitve.

Nastavek za partikularno rešitev:

stopnja $\leq$ stopnja r

$$a_n = n^{\boxed{\alpha}} \boxed{q(n)} \lambda^n$$

→ večkratnost λ v karakteristični polinom. $\alpha \geq 0 \Rightarrow \lambda$ ni ničla k.p.

$$a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} = n - 2 \quad \text{za } n \geq 3$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda-1)^2(\lambda-2) \qquad \lambda = 1$$

1 dvojna
2 enojna

homog.

$$a_n^H = 1^n(A + Bn) + C \cdot 2^n = A + Bn + C \cdot 2^n$$

partikul.

$$a_n^P = n^2(D + En) \cdot 1^n = Dn^2 + En^3$$

$$Dn^2 + En^3 - 4D(n-1)^2 - 4E(n-1)^3 + 5D(n-2)^2 + 5E(n-2)^3$$
$$- 2D(n-3)^2 - 2E(n-3)^3 = n - 2$$

Mogoče je rešitev naslednja $D = -\frac{1}{2}$ $E = -\frac{1}{6}$ D.N.

Končna rešitev:

$$a_n = -\frac{1}{2}n^2 - \frac{1}{6}n^3 + A + Bn + C \cdot 2^n$$