IPMPFMF2025-03-18

Izrek o izomorfizmu:

Let $f: K \to L$ poljuben hom. Ta inducira izo

$$\bar{f}: K/_{\ker f} \longrightarrow \operatorname{Im} f \qquad x \mapsto f(x)$$

s trivialnim jedrom.

kanonična dekompozicija hom. f.

Dodatek:

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "q"'] & L \\
K/_{\ker f} \arrow[r, "\bar{f}"] & \operatorname{Im} f \arrow[u, hook, "i"']
\end{tikzcd}$$

kvocientni hom. q
$x \mapsto [x]$

i (inkluzija)

$\operatorname{Im} f \subseteq L$, $i: \operatorname{Im} f \to L$
$x \mapsto x$

Kaj so ideali v $K/_I$?

Primer: Kaj so ideali v $\mathbb{Z}_{12} = \mathbb{Z}/_{(12)}$?

~~Ups sam prepisujem simbole ¯_(ツ)_/¯ ne vem, kaj se dogaja ... ??!!11 Aaaaaaa wtf ?!11~~

$$\begin{tikzcd}
K \arrow[r, "q"] & K/_I
\end{tikzcd}$$

$J \trianglelefteq_{\text{ideal}}$

$J \longmapsto q_*(J)$

$$q_*(J) = \{ q(a) \,;\, \forall a \in J \} = \{ a + I \,;\, \forall a \in J \}$$

$$\underset{x \in K}{(x+I)} \cdot \underset{a \in J}{(a+I)} = x \cdot a + I \in q_*(J).$$

ideal v K se s q preslika v ideal v K/I.

ZOB kot J ideal v K: $J \xmapsto{q}$ ideal v K/I

$$q_*(J') \longleftarrow J' \triangleleft K/I$$

$\| \; \{a \in K;\ q(a) \in J'\}$ je ideal v K, ki vsebuje I.

$$\left\{\begin{matrix}\text{ideali v } K \text{, ki} \\ \text{vsebujejo } I\end{matrix}\right\} \xleftrightarrow[\text{bijekcija}]{} \left\{\begin{matrix}\text{ideali v} \\ K/I\end{matrix}\right\}$$

IZREK: $q: K \to K/I$ kvocientni homomorfizem določa bijekcijo, ki ideale v K, ki vsebujejo I preslika v ideale v K/I.

$$J \longmapsto q(J)$$

$$q^*(J') \longleftarrow J'$$

↳ preslika

Primer: Kaj so ideali v $\mathbb{Z}_{12} = \mathbb{Z}/(12)$?

To so vsi ideali v $\mathbb{Z}$, ki vsebujejo ideal (12).

v $\mathbb{Z}$ velja $(a) \leq (b) \Leftrightarrow b | a$.

torej so vsi ideali v $\mathbb{Z}_{12}$: $(1)=\mathbb{Z}$, (2), (3), (4), (6), (12).

$$\mathbb{Z}/_{(12)} = \{0\} \qquad \mathbb{Z}/_{(2)} = \{0,2,4,6,8,10\}$$

$$\mathbb{Z}/_{(3)} = \{3,6,9,0\} \qquad \mathbb{Z}/_{(4)} = \{0,4,8\}$$

$$\mathbb{Z}/_{(6)} = \{0,6\} \qquad \mathbb{Z}/_{(1)} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$$

<u>Posledica:</u> K/I je obseg $\iff$

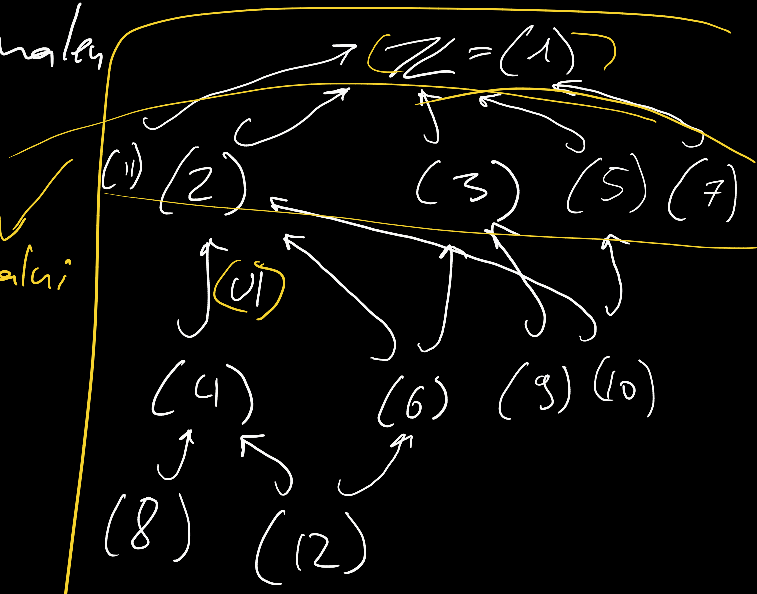
$\iff K/I$ nima pravih idealov $\iff$

$\iff$ v K ni idealov, ki vsebujejo I, razen I in K.

ZDB v K $\nexists$ ideal $J \ni: I \neq J \wedge J \neq K$.

(Pravimo, da je I maksimalen pravi ideal).

$\iff$ I je maksimalen pravi ideal v K. maksimalni



<u>Def:</u> $a \in K$ je nerazcepen, če $\nexists$ neobrnljiva $b,c \ni: a = b \cdot c$

Trditev: V glavnoidealskem kolobarju K je ideal (a) maksimalen $\Leftrightarrow$ a nerazcepen.

Dokaz: $(\Rightarrow)$ let (a) maksimalen in hkrati a razcepen.

$a = b \cdot c$ za neobrnljiva b in c. $\Rightarrow (a) \trianglelefteq (b) \trianglelefteq K$

$\neq$, ker c ni obrnljiv. $\neq$, ker b ni obrnljiv

$\wedge$ $(a) \trianglelefteq (c) \trianglelefteq K$ analogno

stroga pod, stroga pod

$\Rightarrow (a)$ ni maksimalen.

$(\Leftarrow)$ let (a) ni maksimalen $\Rightarrow \exists b$

$(a) \underset{\neq}{\trianglelefteq} (b) \underset{\neq}{\trianglelefteq} K$

$a = b \cdot c$. c ni obrnljiv, ker $(a) \neq (b)$

Primer: $\mathbb{R}[x]/(x^2)$ ni obseg

$(x^2) \to$ razcepen nad $\mathbb{R}[x]$

$\mathbb{R}[x]/(x^2+1)$ je obseg $\mathbb{C}$

$\to$ nerazcepen nad $\mathbb{R}[x]$

[Obsegi]

Primeri: $\mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$

$\mathbb{R} \leq \mathbb{R}(x) \leq (\mathbb{R}(x))(y)$

$\mathbb{C} \leq \mathbb{C}(x)$

$\mathbb{Z}_p \leq \mathbb{Z}_p(x)$

$\mathbb{Q} \underset{\text{strogo}}{<} \{a+b\sqrt{2} \; ; a,b \in \mathbb{Q}\} \underset{\text{strogo}}{<} \mathbb{R}$

dokaz: $a+b\sqrt{2}+c+d\sqrt{2} = a+c+(b+d)\sqrt{2}$ ✓

$$(a+b\sqrt{2})(c+d\sqrt{2}) = ac+ad\sqrt{2}+bc\sqrt{2}+2bd = ac+2bd+(ad+bc)\sqrt{2}$$ ✓

$a+b\sqrt{2}-(c+d\sqrt{2}) = a-c+(b-d)\sqrt{2}$ ✓

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{(a+b\sqrt{2})(c-d\sqrt{2})}{c^2-2d^2} = \frac{ac-2bd}{c^2-2d^2} + \frac{-ad+bc}{c^2-2d^2}\sqrt{2}$$ ✓

zaprtost ✓ $\Rightarrow$ je obseg

$\{a+b\sqrt[3]{4}+c\sqrt[3]{16} \; ; \; a,b,c \in \mathbb{Q}\}$ ... je obseg

$\mathbb{Z}_2$ → glavnoidealski kolobar

$/(x^2+x+1)$ → nerazcepen polinom

je obseg (glej prejšnji izrek)