ANA3VFMF 2025-01-13

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Izračunaj dolžino loka krivulje, podane z

$xz = 2e^{3y}$, $y + \ln 2 = \ln x$ od točke $(2,0,1)$ do točke $(2e,1,e^2)$

$y = t$

$\ln x = t + \ln 2$

$x = e^{t+\ln 2} = e^t \cdot e^{\ln 2} = e^t \cdot 2$

$z = \frac{2e^{3t}}{2e^t} = e^{2t}$

$r(t) = (e^t \cdot 2, t, e^{2t})$ $\quad t \in [0,1]$

$\dot r(t) = (2 \cdot e^t, 1, 2e^{2t})$

dolžina: $\int_0^1 |\dot{\vec r}(t)|\, dt = \int_0^1 \sqrt{4e^{2t}+1+4e^{4t}}\, dt = \int_0^1 \sqrt{(2e^{2t}+1)^2}\, dt =$

$= \int_0^1 2e^{2t}+1\, dt = 2\int_0^1 e^{2t}\, dt + 1 = 2\frac{e^{2t}}{2}\Big|_0^1 + 1 = e^2 - \cancel{e^0} + \cancel{1} = \underline{\underline{e^2}}$

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Ploskev P je podana z enačbo

$z^{2/3} + \underline{(x^2+y^2)}^{1/3} = 1$ — ta ploskev je vrtenina

a) izračunaj površino P

b) izračunaj $\iint_P (x,y,0)\, d\vec S$

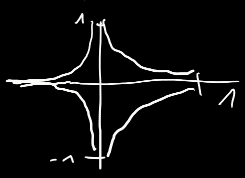
$\rho^2 = x^2 + y^2$

$z^{2/3} + \rho^{2/3} = 1$ $\qquad \rho \geq 0$

VRTENINA:

$\vec r(t,\varphi) = (\rho(t)\cos\varphi, \rho(t)\sin\varphi, z(t))$

najdi ρ, z.



$$z=\cos^3 t \qquad \rho=\sin^3 t \Longrightarrow \cos^2 x+\sin^2 x=1$$

$$t\in[0,\pi]$$

$$Pl(P)=2\pi\int_a^b \rho\sqrt{\dot\rho^2+\dot z^2}\,dt$$

$$\rho(t)=\sin^3 t \qquad z(t)=\cos^3 t \qquad \text{za } t\in[0,\pi]$$

$$\dot\rho(t)=3\sin^2 t\cos t \qquad \dot z(t)=3\cos^2 t(-\sin t)$$

$$Pl(P)=2\pi\int_a^b \rho\sqrt{\dot\rho^2+\dot z^2}\,dt=2\pi\int_0^\pi \sin^3 t\sqrt{9\sin^4 t\cos^2 t+9\cos^4 t\sin^2 t}\,dt=$$

$$=2\pi\int_0^\pi \sin^3 t\sqrt{9\sin^2 t\cos^2 t(\cancel{\sin^2 t+\cos^2 t})}\,dt=2\pi\int_0^\pi 3\sin^4 t\,|\cos t|\,dt=$$

$$=4\pi\int_0^{\pi/2}3\sin^4 t\cos t\,dt=12\pi\int_0^{\frac{\pi}{2}}\sin^4 t\cos t\,dt=12\pi\,\frac{B(\frac{5}{2},1)}{2}=$$

$$=6\pi\,\frac{\Gamma(5/2)}{\Gamma(7/2)}=\frac{6\pi\cdot 2}{5}=\frac{12\pi}{5}$$

b) $\iint_P (x,y,0)\,d\vec S$

$$\vec r(t,\varphi)=(\rho\cos\varphi,\ \rho\sin\varphi,\ z)$$

$$\vec r_t(t,\varphi)=(\dot\rho\cos\varphi,\ \dot\rho\sin\varphi,\ \dot z) \qquad \rho(-\dot z\cos\varphi,\ -\dot z\sin\varphi,\ \dot\rho)$$

$$\|$$

$$\vec r_\varphi(t,\varphi)=(\rho(-\sin\varphi),\ \rho\cos\varphi,\ 0)$$

$$\vec r_t(t,\varphi)\times\vec r_\varphi(t,\varphi)=(-\rho\dot z\cos\varphi,\ -\rho\dot z\sin\varphi,\ \rho\dot\rho\cos^2\varphi+\rho\dot\rho\sin^2\varphi)$$

$$\vec V(\vec r(t))=(\rho\cos\varphi,\rho\sin\varphi,0)=\rho(\cos\varphi,\sin\varphi,0)$$

$$\vec V(\vec r(t))\cdot(\vec r_t\times\vec r_\varphi(t,\varphi))=-\rho^2\dot z\cos^2\varphi-\rho^2\dot z\sin^2\varphi+0=$$

$$=-\rho^2\dot z(\cancel{\cos^2\varphi+\sin^2\varphi})=-\rho^2\dot z$$

$$\iint (x,y,0)\,d\vec{S} = \int_0^{2\pi} d\varphi \int_0^{\pi} -\rho^2 \dot{z}\,dt = 2\pi \int_0^{\pi} \sin^6 t \cdot 3\cos^2 t \sin t\,dt =$$

$$= 6\pi \cdot 2 \int_0^{\pi/2} \sin^7 t \cos^2 t\,dt = 6\pi B(4, 3/2) = 6\pi \frac{\Gamma(4)\,\Gamma(3/2)}{\Gamma(11/2)} =$$

$$= \ldots = 6\pi \frac{3!}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}}$$

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let $f(z) = z^2 - \bar{z}^2$.

a) določi vse ničle f. Ali je f holomorfna?

b) let $[0, 1+i]$ daljica od izhodišča do $1+i$.

izračunaj $\displaystyle\int_{[0,1+i]} f(z)\,dz$.

c) $K = [0,1] \cup [1, 1+i] \cup [1+i, i]$. izračunaj

$$\int_K z^5\,dz$$

a) $z^2 - \bar{z}^2 = (x+iy)^2 - (x-iy)^2 = \cancel{x^2} + 2ixy - \cancel{y^2} - \cancel{x^2} + 2ixy + \cancel{y^2} = 4ixy$

$$4ixy = 0 \iff x = 0 \vee y = 0$$

f holomorfna $\iff u_x = v_y \quad u_y = -v_x$

Princip identičnosti $\{z \in \mathbb{C};\ f(z) = g(z)\}$ s stekališčem

$$\Rightarrow f = g \quad (\text{za } f, g \text{ holomorfni})$$

$\{z \in \mathbb{C};\ f(z) = 0\}$ ima stekališče.

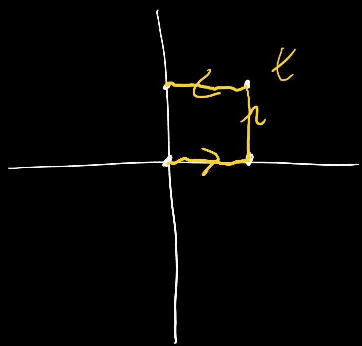
če bi f bila hlmφ, bi bila 0.

toda, ker ni $f(z) = 0$, f ni hlmφ.

b.) $$\int_{[0,1+i]} \underbrace{z^2-\bar{z}^2}_{4ixy}\, dz = \int_0^1 4it^2(1+i)\,dt = 4i(1+i)\int_0^1 t^2\,dt = 4(i-1)\frac{t^3}{3}\Big|_0^1 = 4(i-1)\frac{1}{3} = \frac{4}{3}(i-1)$$

parametrizacija $r(t) = t(1+i)$ $t \in [0,1]$
$\dot r(t) = 1+i$

c.)



$$\int z^5\,dz = \int_{[0,i]} z^5\,dz = \int_0^1 (it)^5\, i\,dt = -\int_0^1 t^5\,dt = -\frac{1}{6}$$

hlmφ, torej $= 0$, torej $= 1$

$r(t) = it$
$\dot r(t) = i$

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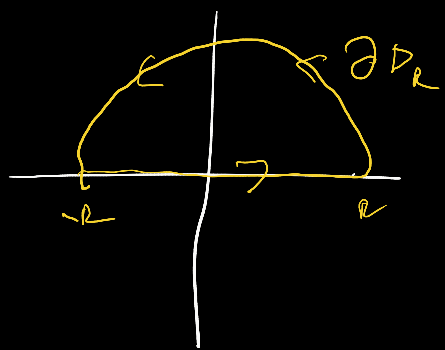
izračunajmo $$\int_{-\infty}^{\infty} \frac{t \sin t}{(t^2+t+1)^2}\,dt =$$ list

$$= \operatorname{Im}\int_{\partial D_R} \frac{z \cdot e^{iz}}{(z^2+z+1)^2}\,dz =$$



singularnosti:

$z^2+z+1 = 0$

$$z_{1,2} = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm\sqrt{i^2 3}}{2} = \frac{-1\pm i\sqrt{3}}{2}$$

$z_1 = \frac{-1+i\sqrt{3}}{2}$, $z_2 = \frac{-1-i\sqrt{3}}{2}$ oba sta stopnji 2.

ogledimo si le tisto singularnost, ki je nad realno osjo: $z_1 = \frac{-1+i\sqrt{3}}{2}$

2. stopnje

$$\operatorname{Res}(f, a) = \lim_{z \to a} \left((z-a)^2 f(z) \right)'$$

$$\operatorname{Res}\left(f, \frac{-1+i\sqrt{3}}{2}\right) = \lim_{z \to \frac{-1+i\sqrt{3}}{2}} \left(\cancel{\left(z - \frac{-1+i\sqrt{3}}{2}\right)^2} \frac{z \cdot e^{iz}}{\cancel{\left(z - \frac{-1+i\sqrt{3}}{2}\right)^2} \left(z - \frac{-1-i\sqrt{3}}{2}\right)^2} \right)' =$$

$$= \lim_{z \to \frac{-1+i\sqrt{3}}{2}} \left(\frac{z \cdot e^{iz}}{z - \left(\frac{-1-i\sqrt{3}}{2}\right)^2} \right)' = \lim_{z \to \frac{-1+i\sqrt{3}}{2}} \left(\frac{z e^{iz} \cdot 4}{4z - 1 - 2i\sqrt{3} + 3} \right)' =$$

$$= \lim_{z \to \frac{-1+i\sqrt{3}}{2}} \left(\frac{4z e^{iz}}{4z - 2i\sqrt{3} + 2} \right)' = \lim_{z \to \frac{-1+i\sqrt{3}}{2}} \left(\frac{2z e^{iz}}{2z - i\sqrt{3} + 2} \right)' = \ldots$$

$$\Rightarrow \oint_{\partial R} f(z)\, dz = 2\pi i \operatorname{Res}\left(f, \frac{-1+i\sqrt{3}}{2}\right)$$