ANA3 VFMF 2025-01-07

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f ima pol $k$-te stopnje v $z=0$.

$$f(z)=a_{-k}z^{-k}+\dots+a_{-1}z^{-1}+a_0+$$

kako izračunati $a_{-1}$ ?

residu

$$z^k f(z)=a_{-k}+\dots+a_{-1}z^{k-1}+a_0z^k+\dots$$

$$\frac{\partial}{(\partial z)^{k-1}} z^k f(z)\Big|_{z=0}=(k-1)!\,a_{-1}+\cancel{k!\,a_0}^{\,0}+\dots$$

$$\operatorname{Res}(f,a)=\lim_{z\to a}\frac{1}{(k-1)!}\left(\frac{d}{(dz)^{k-1}}(z-a)^k f(z)\right)$$

če ima f pol stopnje 1:

$$\operatorname{Res}(f,a)=\lim_{z\to a}(z-a)f(z)$$

f pol stopnje 2:

$$\operatorname{Res}(f,a)=\lim_{z\to a}\left(\frac{d}{dz}\left((z-a)^2f(z)\right)\right)$$

1.) izračunaj $\operatorname{Res}\left(\frac{1}{(1+z^2)^2},i\right)$ singularnosti $\pm i$, tipa pol stopnje 2.

ampak recimo, da ne vemo stopnje pola. postopamo s stopnjo 1 (napačno):

$$\operatorname{Res}(\dots, i) = \lim_{z \to i} (z-i) \frac{1}{(1+z^2)^2} \overset{L.H.}{=} \lim_{z \to i} \frac{(z-i)'}{((1+z^2)^2)'} =$$

$$= \lim_{z \to i} \frac{1}{2(1+z^2)2z} = \lim_{z \to i} \frac{1}{4z(1+z^2)} \overset{\text{vstavi } i=z}{=} \frac{1}{4i(0)} = \text{undef}$$

ni sicer večeno, da je tudi izvorna f-ja undef (po L.H.), ampak tu se izkaže, da je tudi izvorna f-ja undef.

Ker je undef $\Rightarrow$ <u>napačna izbira stopnje pola</u>

pol ima v resnici stopnjo 2:

$$\operatorname{Res}(\dots, i) = \lim_{z \to i} \left( \left( (z-i)^2 \frac{1}{(1+z^2)^2} \right)' \right) =$$

$$= \lim_{z \to i} \left( \frac{\cancel{(z-i)^2}}{(\cancel{(z-i)}(z+i))^2} \right)' = \lim_{z \to i} \left( \frac{1}{(z+i)^2} \right)' = \lim_{z \to i} (-2)(z+i)^{-3} =$$

$$= \lim_{z \to i} \frac{-2(z+i)}{(z+i)^4} \overset{L.H.}{=} \lim_{z \to i} \frac{-2}{(z+i)^3} \overset{\text{vstavi } z=i}{=} \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = -\frac{1}{4} i$$

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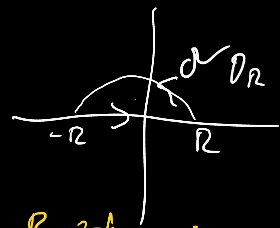
izračunaj $\displaystyle\int_0^\infty \frac{1}{1+t^4}\,dt = \frac{1}{2} \int_{-\infty}^\infty \frac{dt}{1+t^4}$

definiramo $f(z)=\frac{1}{1+z^4}$ in integriramo po robu polkroga za $R\in\mathbb{R}$ (poljubno velik) $R>0$ $R\gg 1$

$$\oint_{\gamma D_R} \frac{1}{1+z^4}\,dz = 2\pi i \sum_{a_i\in D_R} \mathrm{Res}\left(\frac{1}{1+z^4}, a_i\right)$$



→ za dovolj velik $R$ zajamemo vse singularnosti in za večji $R$ ne spremenimo vrednosti integrala, tako ne rabimo lim za dovolj velike $R$, t.j. $R\gg 1$

singularnosti $\frac{1}{1+z^4}$ $\qquad 1+z^4=0$

$$e^{\pi i + 2k\pi i} = -1 = z^4, \quad k\in\mathbb{Z}$$

kompleksni pari, ker je polinom realen

$$z^4=-1=e^{\pi i+2k\pi i}$$

$$z_{1,2,3,4}=(-1)^{1/4}=e^{\frac{\pi i}{4}+\frac{k\pi i}{2}} = \begin{cases} e^{\frac{\pi i}{4}} \\ e^{\frac{3\pi i}{4}} \\ e^{\frac{5\pi i}{4}} = e^{\frac{-\pi i}{4}} \\ e^{-\frac{3\pi i}{4}} \end{cases} =$$

$$= \begin{cases} \frac{1+i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} \\ \frac{-1+i}{\sqrt{2}} \\ \frac{-1-i}{\sqrt{2}} \end{cases}$$

ležita v $D_R$

$$\lim_{R\to\infty}\oint_{\gamma D_R} \frac{1}{1+z^4}\,dz = \underbrace{\lim_{R\to\infty}\int_{[-R,R]} \frac{1}{1+z^4}\,dz}_{\int_{-\infty}^{\infty}\frac{1}{1+t^4}dt} + \underbrace{\lim_{R\to\infty}\int_K \frac{1}{1+z^4}\,dz}_{0} =$$

integral iz naloge!

$$= \int_{-\infty}^{\infty} \frac{1}{1+t^4}\,dt$$

$$\left| \int_{\mathcal{L}} \frac{1}{1+z^4}\,dz \right| \leq \int_{\mathcal{L}} \frac{1}{|1+z^4|}\,|dz| \approx$$

$$\approx \frac{1}{R^4} \cdot \pi R \rightarrow 0$$

$|z| = R$ na $\mathcal{L}$

$$\rightsquigarrow \frac{1}{|1+z^4|} \approx \frac{1}{R^4}$$

izračunajmo residue:

$$\oint_{\partial D_R} \frac{1}{1+z^4}\,dz = i2\pi \underbrace{\sum_{r \in \left\{\frac{1+i}{\sqrt{2}},\ \frac{-1+i}{\sqrt{2}}\right\}} \operatorname{Res}\left(\frac{1}{1+z^4}, r\right)}_{\text{residua na } D_R} = \dots$$

$$\operatorname{Res}\left(\frac{1}{1+z^4}, \frac{1+i}{\sqrt{2}}\right) = \lim_{z \to \frac{1+i}{\sqrt{2}}} \left(z - \frac{1+i}{\sqrt{2}}\right) \frac{1}{1+z^4} =$$

$$\overset{L'H}{=} \lim_{z \to \frac{1+i}{\sqrt{2}}} \frac{1}{4z^3} = \frac{1}{4\,\frac{-1-i}{\sqrt{2}}}$$

podobno za drugi residu

$$\dots = i2\pi\left(\frac{1}{4\,\frac{-1-i}{\sqrt{2}}} + \frac{1}{4\,\frac{1-i}{\sqrt{2}}}\right) = \dots \in \mathbb{R}\ \checkmark$$

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izračunaj $\displaystyle\int_0^\infty \frac{\cos t}{1+t^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos t}{1+t^2}\,dt = \frac{1}{2}\operatorname{Re}\int_{\partial D_R}\frac{e^{iz}}{1+z^2}\,dz =$

$$= \frac{1}{2}\operatorname{Re}\Big(2\pi i \sum_{\substack{\text{singularn.}\\ a_i \in D_R}} \operatorname{Res}\Big(\frac{e^{iz}}{1+z^2}, a_i\Big)\Big) = \frac{1}{2}\operatorname{Re}\Big(2\pi i\,\frac{e^{-1}}{2i}\Big) = \frac{1}{2}\operatorname{Re}\Big(\cancel{2}\pi\,\frac{e^{-1}}{\cancel{2}}\Big) = \frac{\pi}{2e}$$

kot v prejšnji nalogi

singularnosti $\dfrac{e^{iz}}{1+z^2}$ : $z_{1,2} = \pm i$

edina v $D_R$ je $a = i$

$$\operatorname{Res}\Big(\frac{e^{iz}}{1+z^2}, i\Big) = \lim_{z\to i}(z-i)\frac{e^{iz}}{1+z^2} = \lim_{z\to i}\frac{\cancel{(z-i)}e^{iz}}{\cancel{(z-i)}(z+i)} =$$

$$= \lim_{z\to i}\frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$$

zakaj smo ta vzeli $e^{iz} = \cos z$ ?

$$\int_{K_R}\frac{e^{z}}{1+z^2}\,dz \xrightarrow{??} 0$$

$$|e^{iz}| = |e^{-y}e^{ix}| = e^{-y} = e^{-\operatorname{Im} z} \le 1$$

kar pa ne velja za $|\cos z|$