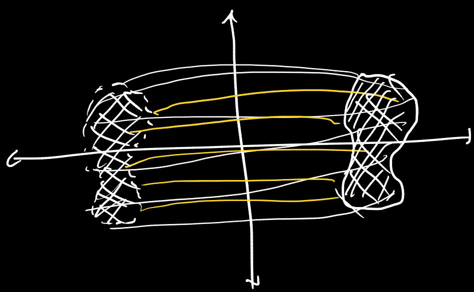
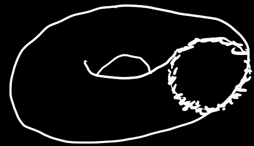
ANA3VFMF 2024-12-16

[Vrtenine]



npr. torus



medtem ko je volumen vrtenine preprost, je površina malce manj intuitivna.

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let K $t \longmapsto (\rho(t), z(t))$ $\qquad$ ($\rho \geq 0 \sim K \in [0,\infty) \times \mathbb{R}$)

zapiši integral, ki določa površino vrtenine, ki jo dobimo tako, da K zavrtimo okoli z-osi.

cilindrične: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$ $\qquad$ opazka: $r = \rho(t)$, $z = z(t)$

$$\vec{r}(\varphi, t) = (\rho(t)\cos\varphi, \rho(t)\sin\varphi, z(t))$$

, definicijsko območje:

$$D = (a, b)_t \times (0, 2\pi)_\varphi$$

$$\underbrace{pl(P) = \iint_{D=(a,b)_t\times(0,2\pi)_\varphi} |\vec{r}_t \times \vec{r}_\varphi| \, dt\, d\varphi = 2\pi \int_a^b \rho(t)\cdot\sqrt{\dot{z}^2(t) + \dot{\rho}^2(t)}\, dt}_{\text{formula za površino vrtenine}}$$

$$\vec{r}_t = (\dot\rho(t)\cos\varphi, \dot\rho(t)\sin\varphi, \dot z(t))$$
$$\vec{r}_\varphi = (-\rho(t)\sin\varphi, \rho(t)\cos\varphi, 0)$$
$$\vec{r}_t \times \vec{r}_\varphi = \left(-(\dot z\rho)(t)\cos\varphi, -(\dot z\rho)(t)\sin\varphi, \dot\rho(t)\rho(t)\cos^2\varphi + \dot\rho(t)\rho(t)\sin^2\varphi = \dot\rho(t)\rho(t)\right)$$
$$|\vec{r}_t \times \vec{r}_\varphi| = \sqrt{\dot z^2(t)\rho(t) + \dot\rho^2(t)\rho^2(t)}$$

b) izračunaj površino ploskve $(x^2+y^2)^{3/2} + z = 1$; $z \geq 0$

to je vrtenina, ker je odvisna od x^2+y^2 (rotacijska simetričnost)

K če trivialno, ti še dobimo, ko vstavimo $y=0$ in $x=\rho$:

$\rho^2 = x^2+y^2$, $\quad \rho^{4/3} + z = 1$, $\quad z = 1-\rho^{4/3}$

parametrizacija: $t \longmapsto (t, 1-t^{4/3})$; $\quad t \in [0,1]$

$$pl(P) = 2\pi \int_0^1 t \cdot \sqrt{\left(\frac{4}{3} t^{1/3}\right)^2 + 1}\, dt = 2\pi \cdot \frac{1}{3} \int_0^1 t \sqrt{16 t^{2/3} + 9}\, dt =$$

$u = t^{2/3}$

$du = \frac{2}{3} t^{-1/3}\, dt$

$dt = \frac{3}{2} t^{1/3}$

$$= \pi \int_0^1 u^2 \sqrt{16u+9}\, du =$$

$w = 16u+9$

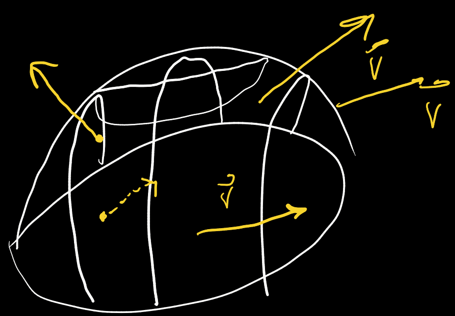
$u = \frac{w-9}{16}$

$$= \dots \int_0^1 (w-9)^2 \sqrt{w}\, dw = \dots$$

D.N.

D.N. površina torusa

$\iint_P \vec{v}\, d\vec{S}$ = pretok



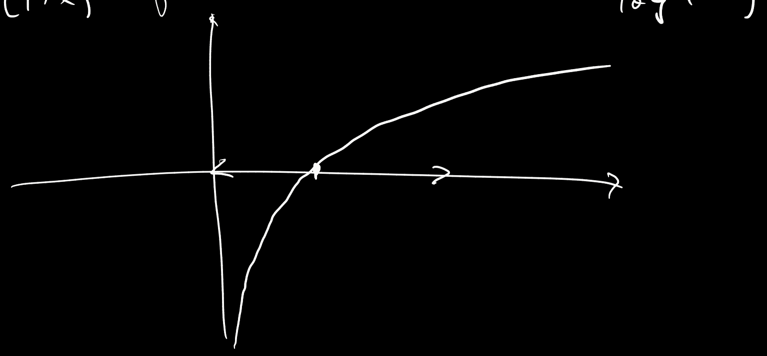
asistent na vajah pokaže youtube short od dnote-digitalguide.

D.N.: dodatnih 7% do S.1.

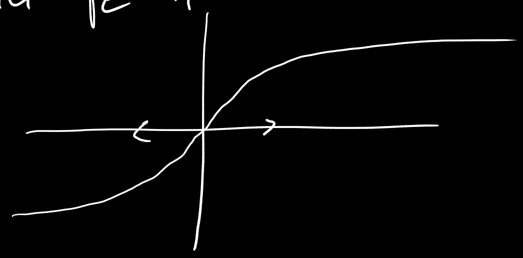
N [Kompleksna števila]

Taylor za arctan $x = x + \dots$

konvergenčni radij $\log(1+x)$ je 1



konvergenčni radij atan je 1



zakaj? ker atan ima pol v $x=i$ (na razdalji 1)

zakaj?

oglejmo si odvod: $\operatorname{atan}' x = \frac{1}{1+x^2}$; ima pol v $x=i$ očitno

$\Rightarrow$ konvergenčni radij se vedno ustavi pri polu, ta pol je lahko tudi kompleksen.

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za $n \in \mathbb{N}$ izračunaj vsoto $\sum_{k=0}^{n} \cos(k)$. √√√

$$\sum_{k=0}^{n} \cos(k) = \sum_{k=0}^{n} \mathrm{Re}(e^{ik}) = \mathrm{Re}\Big(\sum_{k=0} e^{ik}\Big) = \quad -\cancel{2i}\sin\left(\frac{n+1}{2}\right)$$

$$= \mathrm{Re}\,\frac{1-e^{i(n+1)}}{1-e^{i}} = \mathrm{Re}\left(\frac{e^{\frac{i(n+1)}{2}}\left(e^{-\frac{i(n+1)}{2}} - e^{\frac{i(n+1)}{2}}\right)}{e^{i/2}\left(e^{-i/2} - e^{i/2}\right)}\right) =$$

$\cancel{\cos\left(\frac{1}{2}\right)} + i\sin\left(\frac{1}{2}\right) \cancel{- \cos\left(\frac{1}{2}\right)} - i\sin\left(\frac{1}{2}\right) = \cancel{-2}i\left(\frac{1}{2}\right)$

$$= \mathrm{Re}\left(e^{\frac{i(n+1)}{2} - \frac{i}{2}} \cdot \frac{\sin\frac{n+1}{2}}{\sin 1/2}\right) =$$

$$= \frac{\sin\frac{n+1}{2}}{\sin 1/2} \,\mathrm{Re}\; e^{i\frac{n}{2}} = \frac{\sin\frac{n+1}{2}}{\sin 1/2} \cdot \cos\frac{n}{2}$$

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kompleksne f/e

$f: \mathbb{C} \to \mathbb{C}$ $\qquad u = \mathrm{Re} f$

$\quad x+iy \qquad\qquad v = \mathrm{Im} f$

$f(x+iy) = u(x,y) + iv(x,y)$

$f(z) = u(\mathrm{Re} z, \mathrm{Im} z) + iv(\mathrm{Re} z, \mathrm{Im} z)$

N

Zapiši f v kompleksnih argumentih

z in $\bar{z}$ in za dan $D \subset \mathbb{C}$ določi $f(D)$

a.) $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$, $D = \{0 \leq \arg z \leq \frac{\pi}{4}\}$

$f(x+iy) = x^2 - y^2 + 2xyi = (x+iy)^2$

$f(z) = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 + 2\,\frac{(z+\bar{z})(z-\bar{z})}{4} =$

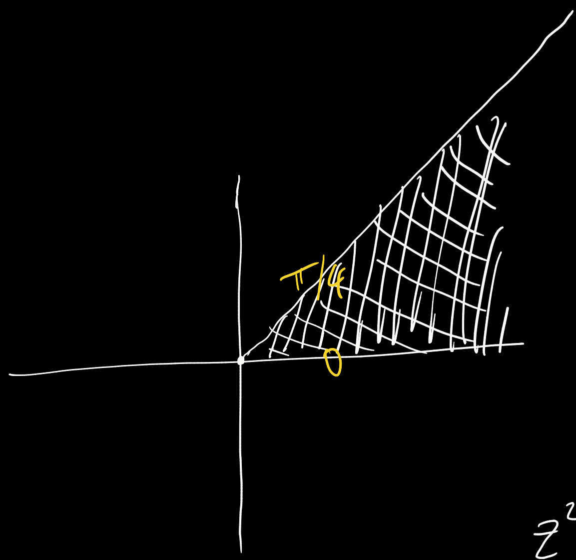
$z = x+iy$

$x = \frac{z+\bar{z}}{2}$

$y = \frac{z-\bar{z}}{2i}$

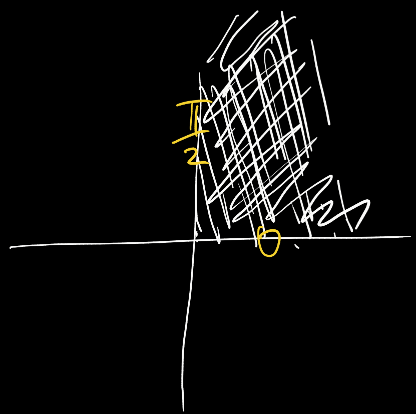
$= \dots = z^2$

$D:$



$z^2 = (Ae^{i\varphi})^2 = A^2 e^{i2\varphi}$

$f(D):$



b) $u(x,y) = x^2 + y^2$, $v(x,y) = 2xy$, $D = \{\operatorname{Im} z > 0\}$

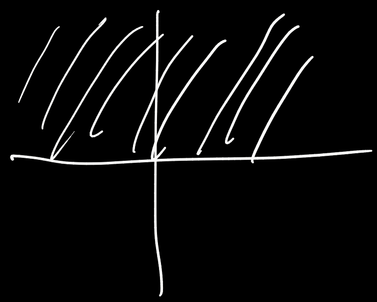
$f(x+iy) = x^2 + y^2 + 2xy$

$\left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 + 2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) i =$

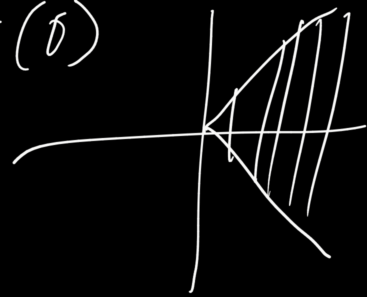
$= \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} - 4\,\frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \cancel{2}\,\frac{z^2 - \bar{z}^2}{\cancel{4}^2} =$

$$= \frac{\cancel{4} z\bar{z}}{\cancel{4}} + \frac{z^2+\bar{z}^2}{2}$$

D:



$f(D)$



N

Na dva različita načina izračunaj

$\int_\gamma e^z\,dz$, gdje je γ dužina od 0 do $1+i$,

parametrizacija $\gamma(t) = (t,t) = t(1+i)$,

$t \in (0,1)$

def:

$$\int_\gamma e^z\,dz := \int_0^1 e^{\overbrace{t(1+i)}^{\gamma(t)}} \cdot \underbrace{(1+i)\,dt}_{\dot\gamma(t)\,dt} =$$

$z = t(1+i)$

$dz = (1+i)\,dt$

kompleksno množenje

$$= \int_0^1 e^t(\cos t + i\sin t)(1+i)\,dt =$$

$$= \int_0^1 e^t(\cos t - \sin t) + e^t i(\sin t + \cos t)\,dt = $$

$$= \int_0^1 e^t (\cos t - \sin t)\,dt + i\int_0^1 e^t (\sin t + \cos t)\,dt$$

drugi nacin

$$\int_{\gamma} e^z\,dz = e^z \Big|_0^{1+i}$$

$\gamma = [0, 1+i]$

dalzica od 0 do $1+i$

? zasto to vrijedi?

D.N.