ANA3 VFMF 2024-12-08

meta:
Vaje: tor, 7.1., od 3h–5h
Preostali tvizi bodo kot DN:
okoli 17.12. bo prva.

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naj bo P del paraboloida $z = 1 - x^2 - y^2$ nad xy ravnino $(z \geq 0)$.

a) Parametriziraj P in določi tangentno ravnino v vsaki točki.

$$D = \{u^2 + v^2 \leq 1\}$$

$$\vec{r}(u,v) = (u, v, 1 - u^2 - v^2)$$

V 2D imamo 2 odvoda, $\vec{r}_u$ in $\vec{r}_v$.
Parametrizacija je gladka, če $|\vec{r}_u \times \vec{r}_v| \neq 0$

$$\vec{r}_u = (1, 0, -2u)$$
$$\vec{r}_v = (0, 1, -2v)$$

$$\vec{r}_u \times \vec{r}_v = (2u, 2v, 1) \neq \vec{0}, \text{ ker } \ldots, 1).$$

tangentna ravnina skozi (x_0, y_0, z_0) z normalo $\vec{n}_{(x_0,y_0,z_0)} = \vec{r}_u \times \vec{r}_v$

$$T_{(x_0,y_0,z_0)} = \{(x,y,z) \in \mathbb{R}^3 ; [(x,y,z) - (x_0,y_0,z_0)] \cdot \vec{n}_{(x_0,y_0,z_0)} = 0\}$$

$$\vec{n}_{(x_0,y_0,z_0)} = (2x_0, 2y_0, 1)$$

b) Izračunaj ploščino (površino) ... $pl(P)$.

$$pl(P) = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\,dv = \iint_D \sqrt{4u^2 + 4v^2 + 1} \, du\,dv = \quad \text{(polarne)}$$

$$= \int_0^{2\pi} d\varphi \int_0^1 \sqrt{4r^2+1}\, r\,dr = \frac{2}{8}\pi \cdot \int_1^5 \sqrt{t}\, dt = \frac{\pi}{4} \cdot \frac{2t^{\frac{3}{2}}}{3} \Bigg|_1^5$$

$4r^2 + 1 = t$
$8r\,dr = dt$

c.) Naj K neka krivulja, ki leži na ploskvi P. Izračunaj

$$\int_K x\,dx + y\,dy + \tfrac{1}{2}\,dz =$$

$$= \int_K \left(x, y, \tfrac{1}{2}\right) d\vec{s} =$$

$\vec{s}: (a,b) \to K$ parametrizacija

$\vec{s}(t) = (x(t), y(t), z(t))$

$\forall t \in (a,b): (x(t), y(t), z(t)) \in K$

$$= \int_a^b \underbrace{\left(x(t), y(t), \tfrac{1}{2}\right)}_{\to (x,y,\frac{1}{2}) = \vec{n}(x(t),y(t),z(t))} \cdot \dot{\vec{s}}\,dt = 0$$

$\dot{\vec{s}}$ je tangenta K in, ker $K \subseteq P$, je $\dot{\vec{s}} \perp \vec{n}$.

velja $\vec{s}(t) = \vec{r}(u(t), v(t))$

$\dot{\vec{s}} = \vec{r}_u \dot{u} + \vec{r}_v \cdot \dot{v} \in$ tangentna ravnina

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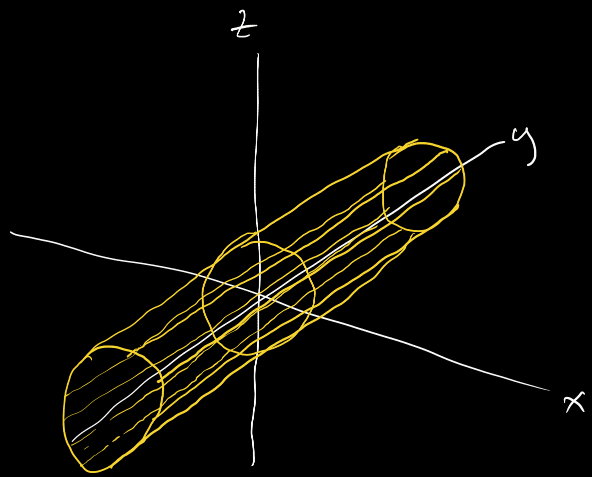
Plašč valja P je podan z enačbo $x^2 + z^2 = 1$. Parametriziraj P in izračunaj $\iint_P x^2 z^4 e^{-y^2}\,dS$ ter $\iint_P (0, y, 0)\,d\vec{S}$.



$\vec{r}(\varphi, y) = (\cancel{r}\cos\varphi, y, \cancel{r}\sin\varphi) =$

$= (\cos\varphi, y, \sin\varphi)$

$D = [0, 2\pi)_\varphi \times (-\infty, \infty)_y$

$$\iint_P x^2 z^4 e^{-y^2}\,dS = \iint_D \cos^2\varphi \sin^4\varphi\, e^{-y^2} \cdot \cancel{|\vec{r}_\varphi \times \vec{r}_y|}\,d\varphi\,dy =$$

$$= \iint_D \cos^2\varphi \sin^4\varphi\, e^{-y^2}\,d\varphi\,dy \overset{\text{fubini}}{=\!=}$$

$$= \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} \cos^2\varphi \sin^4\varphi\, e^{-y^2}\,dy = \ldots$$

$\vec{r}_\varphi = (-\sin\varphi, 0, \cos\varphi)$

$\vec{r}_y = (0, 1, 0)$

$|\vec{r}_\varphi \times \vec{r}_y| = |(-\cos\varphi, 0, -\sin\varphi)| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$

$$\dots = \left(\int_0^{2\pi} \cos^2\varphi \, \sin^4\varphi \, d\varphi\right)\left(\int_{-\infty}^{\infty} e^{-y^2} \, dy\right) =$$

↓ samo še enkrat prej

$$= \left(4\int_0^{\frac{\pi}{2}} \cos^2\varphi \, \sin^4\varphi \, d\varphi\right)\left(\sqrt{\pi}\right) = 2B\left(\tfrac{3}{2}, \tfrac{5}{2}\right)\cdot\sqrt{\pi}$$

$$\iint_P (0,y,0)\, d\vec{S} = \iint_D (0,y,0)\cdot \underbrace{(\vec{r}_\varphi \times \vec{r}_y)\, d\varphi\, dy}_{d\vec{S}} = \iint_D (0,y,0)\cdot(-\cos\varphi, 0, -\sin\varphi)\, d\varphi\, dy =$$

$$= \iint_D 0 \, d\varphi\, dy = 0$$

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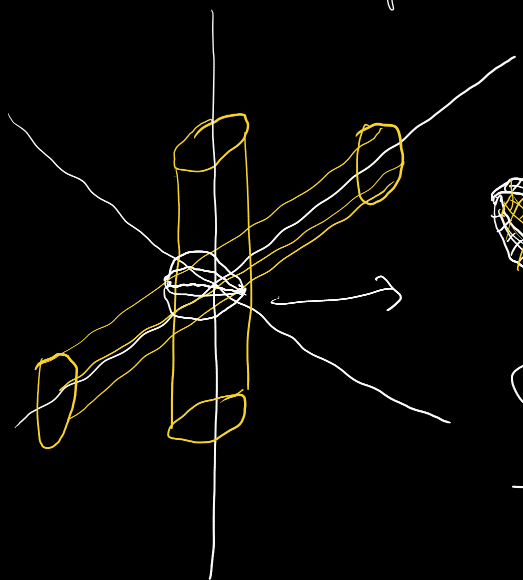
let P plašč telesa, ki ga omejujeta ploskvi $x^2+y^2=1$ in $x^2+z^2=1$.

Izračunaj površino P in $\iint_P (0,y,0)\, d\vec{S}$.



parametriziramo:

$\vec{r} = (\cos\varphi, y, \sin\varphi)$

toda ne teče po celem valju, ampak le po enem kosu

vstavimo v $x^2+y^2=1$:

$\cos^2\varphi + y^2 = 1$

$y^2 = 1-\cos^2\varphi$

$y^2 = \sin^2\varphi \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \pm \sin\varphi \qquad y \in [-\sin\varphi, \sin\varphi]$

$$pl(P) = 4\iint_D |\vec{r}_\varphi \times \vec{r}_y|\, d\varphi\, dy =$$

$D = \{(\varphi, y);\ \varphi \in [0,\pi],\ -\sin\varphi \le y \le \sin\varphi\}$

$$= 4\int_0^{\pi} d\varphi \int_{-\sin\varphi}^{\sin\varphi} dy = 4\int_0^{\pi} y \Big|_{-\sin\varphi}^{\sin\varphi} d\varphi =$$

$$= 8\int_0^{\pi} \sin\varphi\, d\varphi = -8\cos\varphi\Big|_0^{\pi} = 8+8 = 16$$

$$\iint_P (0,y,0)\,d\vec{S} = \sum_{i=1}^{4} \iint_{P_i} (0,y,0)\,d\vec{S} = 2\iint_{P_3} (0,y,0)\,d\vec{S} = \ldots$$

$\downarrow$

za $i=1$ in $i=2$ je
integral po P_1 in $P_2 = 0$. (naloga 2)

$$\ldots = \iint_{D'} (0,y,0)(\vec{r}_\varphi \times \vec{r}_z)\,d\varphi\,dz = 2\iint_{D'} (0,y,0)\cdot(\cos\varphi, \sin\varphi, 0)\,d\varphi\,dz =$$

$$= \iint_{D'} \sin^2\varphi\,d\varphi\,dz = 2\int_0^{\pi} d\varphi \int_{-\sin\varphi}^{\sin\varphi} \sin^2\varphi\,dz = 2\int_0^{\pi} \sin^2\varphi \cdot z\Big|_{-\sin\varphi}^{\sin\varphi}\,d\varphi =$$

$$= 4\int_0^{\pi} \sin^3\varphi\,d\varphi = 8\int_0^{\pi/2} \sin^3\varphi\,d\varphi = 4B\left(2,\tfrac{1}{2}\right) = \ldots = Vol\,(\text{telesa})$$