ANA3VFMF 2024-11-18 Kolokvij: 29.11. @ 16:00

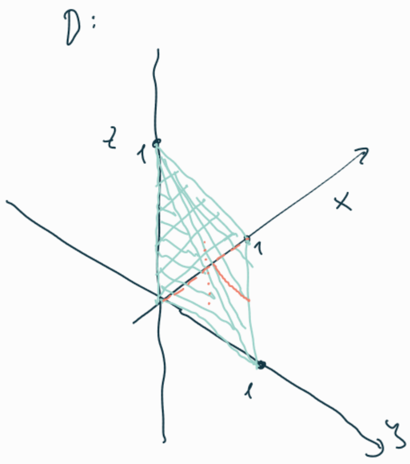
N

let D območje, omejeno z: $x+y+z=1$

$x=0,\ y=0,\ z=0$

izračunaj $\iiint\limits_D \frac{1}{(1+x+y+z)^3}\,dx\,dy\,dz$!

D:



fubini $\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\,dz\,dy\,dx =$

$$= \int_0^1\int_0^{1-x} \frac{-1}{2(1+x+y+z)^2}\Big|_0^{1-x-y} dy\,dx =$$

$$= \int_0^1\int_0^{1-x} \frac{-1}{2(1+\cancel{x+y}+1\cancel{-x-y})^2} - \frac{-1}{2(1+x+y)^2}\,dy\,dx = \int_0^1\int_0^{1-x} -\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{(1+x+y)^2}\,dy\,dx =$$

$$= \int_0^1\int_0^{1-x} -\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{(1+x+y)^2}\,dy\,dx = \int_0^1 \left(-\frac{y}{8} + \frac{1}{2}\cdot\frac{-1}{1+x+y}\right)\Big|_0^{1-x} dx =$$

$$= \int_0^1 \left(-\frac{1-x}{8} + \frac{1}{2}\cdot\frac{-1}{1+\cancel{x}+1\cancel{-x}} + \frac{1}{2}\cdot\frac{1}{1+x}\right)dx =$$

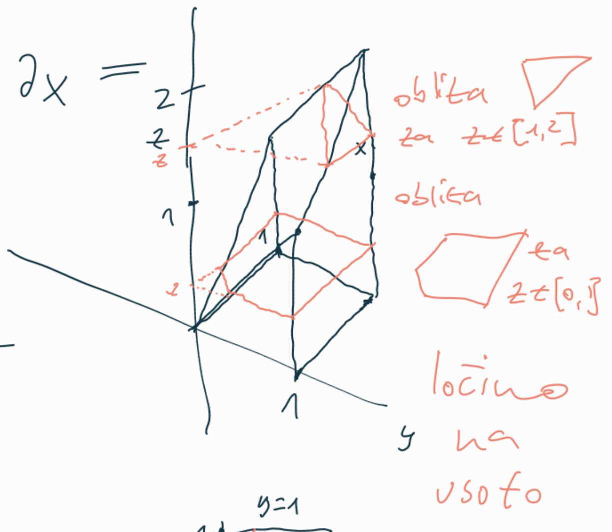
$$= \int_0^1 \frac{x-1}{8} - \frac{1}{4} + \frac{1}{2(1+x)}\,dx = \left(\frac{(x-1)^2}{16} - \frac{x}{4} + \frac{1}{2}\ln|1+x|\right)\Big|_0^1 =$$

$$= -\frac{1}{16} - \frac{1}{4} - \frac{1}{2}\ln 2 \cancel{- \frac{1}{2}\ln|1|}$$

N

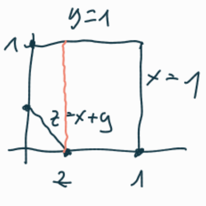
zamenjaj vrstni red integracije tako, da „ostane" le enojni integral!:

$$\int_0^1\int_0^1\int_0^{x+y} f(z)\,dz\,dy\,dx =$$



$$= \underbrace{\int_1^2\int_{z-1}^1\int_{z-x}^1 f(z)\,dy\,dx\,dz}_{\text{izračunajmo le tole}} +$$

$$+ \underbrace{\int_0^1 \cdots dz}_{\text{za dn}}$$



$$\int_1^2\int_{z-1}^1\int_{z-x}^1 f(z)\,dy\,dx\,dz =$$

$$= \int_1^2\int_{z-1}^1 f(z)\,y\Big|_{z-x}^1 \,dx\,dz = \int_1^2\int_{z-1}^1 f(z)\,(1-z+x)\,dx\,dz =$$

$$= \int_1^2 f(z)x - f(z)zx + f(z)\frac{x^2}{2}\Big|_{z-1}^1 \,dz =$$

$$= \int_1^2 f(z) - zf(z) + \frac{1}{2}f(z) - f(z)(z-1) + f(z)z(z-1) - f(z)\frac{(z-1)^2}{2}\,dz =$$

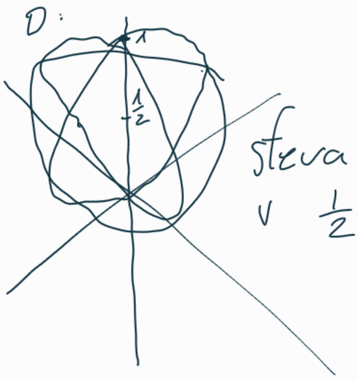
$$= \int_1^2 f(z)\left(1 - z + \frac{1}{2} - (z-1) + z(z-1) - \frac{1}{2}(z-1)^2\right)dz$$

N

Izračunaj $\iiint_D \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ za $D := \{(x,y,z)\in\mathbb{R}^3 ; \boxed{x^2+y^2+z^2\le z}\}$

$x^2+y^2+z^2-z\le 0$

$x^2+y^2+(z-\frac{1}{2})^2\le(\frac{1}{2})^2$



sfera s središčem v $\frac{1}{2}$ in radijem $\frac{1}{2}$

sferično

$r^2\le r\cdot\sin\theta$

$r\le\sin\theta$

$\Rightarrow$ sferične koordinate

$x = r\cos\varphi\cos\theta$

$y = r\sin\varphi\cos\theta$

$z = r\sin\theta$

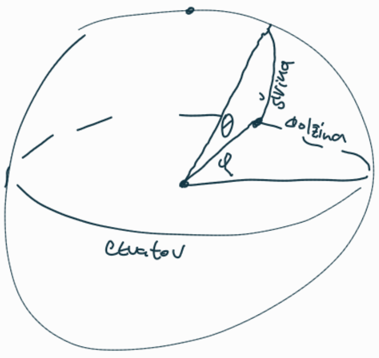
$|\det J| = r^2\cos\theta$

$r = \sqrt{x^2+y^2+z^2}$

$r\in[0,\infty)$

$\varphi\in[0,2\pi)$

$\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$



$$= \iiint \underbrace{r}_{\text{fja}}\,\underbrace{r^2\cos\theta}_{\text{jac}}\,dr\,d\varphi\,d\theta =$$

$\Omega = \{(r,\varphi,\theta) ; r\le\sin\theta,\ \varphi\in[0,2\pi),\ \theta\in(-\frac{\pi}{2},\frac{\pi}{2}),\ r\ge 0\}$

fubini $\int_{\boxed{0}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\sin\theta} \underbrace{r^3\cos\theta}_{\text{ni odvisnosti od } \varphi}\, dr\, d\varphi\, d\theta = 2\pi \int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r^3\cos\theta\, dr\, d\theta =$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\cos\theta \Big|_{r=0}^{\sin\theta} d\theta = 2\pi \int_0^{\frac{\pi}{2}} \frac{\sin^4\theta}{4}\cos\theta\, d\theta = \frac{\pi}{2}\cdot\frac{1}{2} B\left(\frac{5}{2},1\right) =$$

$2p-1=4 \Rightarrow p=\frac{5}{2}$

$2q-1=1 \Rightarrow q=1$

$$= \frac{\pi}{4} B\left(\frac{5}{2},1\right) = \frac{\pi}{4}\frac{\Gamma(\frac{5}{2})\Gamma(1)}{\Gamma(\frac{7}{2})} =$$

$$= \frac{\pi}{4}\frac{\cancel{\Gamma(\frac{5}{2})}}{\frac{5}{2}\cdot\cancel{\Gamma(\frac{5}{2})}} = \frac{\pi}{10}$$

N

Izračunaj volumen območja $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} \leq 1$

za $a,b,c>0$.

$\downarrow$ vpeljemo nove spremenljivke:

$u=\sqrt[3]{\frac{x}{a}} \quad v=\sqrt[3]{y/b} \quad w=\sqrt[3]{z/c}$

$x=au^3 \quad y=bv^3 \quad z=cw^3$

$$V=\iiint_D dx\,dy\,dz =$$

$$= \iiint \underbrace{1}_{f(\ldots)} \underbrace{27abc\,u^2v^2w^2}_{\text{jacobi}}\, du\,dv\,dw =$$

$\Omega = \{(u,v,w)\in\mathbb{R}^3;\ u^2+v^2+w^2 \leq 1\}$

vpeljimo sferične koordinate: $u=r\cos\varphi\cos\theta \quad v=r\sin\varphi\cos\theta \quad w=r\sin\theta$

$$\det J = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = \begin{vmatrix} 3au^2 & 0 & 0 \\ 0 & 3bv^2 & 0 \\ 0 & 0 & 3cw^2 \end{vmatrix} = 27abc\,u^2v^2w^2$$

$$= \int_0^1\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 27abc\, r^2\cos^2\varphi\cos^2\theta\, r^2\sin^2\varphi\cos^2\theta\, r^2\sin^2\theta \cdot \underbrace{r^2\cos\theta}_{\det J}\, d\theta\, d\varphi\, dr =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 27abc\, r^8\cos^2\varphi\sin^2\varphi\cos^5\theta\sin^2\theta\, d\theta\, d\varphi\, dr =$$

$$= 27abc \underbrace{\int_0^1 r^8\, dr}_{1/3} \underbrace{\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\, d\varphi}_{4\cdot\int_0^{\frac{\pi}{2}}\ldots d\varphi = 2B(\ldots)} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\theta\sin^2\theta\, d\theta}_{2\int_0^{\frac{\pi}{2}}\ldots d\theta = B(\ldots)} = \ldots \text{D.N.}$$