ANA3VFMF2024-11-11 meta 19.11.2024 15h

2.03

N

Izračunaj volumen ned grafom f/e $f(x,y)=1-x^2-y^2$ in ravnino $z=0$:

$x=r\cos\varphi$ $y=r\sin\varphi$ $x^2+y^2=r^2$ $J=\begin{vmatrix} x_r & y_r \\ x_\varphi & y_\varphi \end{vmatrix}=r$

$$\iint_D \boxed{1\overbrace{-x^2-y^2}^{-r^2}}\,dx\,dy = \iint_\Omega \underbrace{(1-r^2)}_{\text{fja}}\cdot\underbrace{r}_{\text{Jacobi}}\,dr\,d\varphi \quad \text{fubini}$$

$D=\{(x,y);\ x^2+y^2\le 1\}$

Ω = $r\in[0,1]$, $\varphi\in[0,2\pi]$

$$=\int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\varphi = \int_0^{2\pi}\left(\int_0^1 r\,dr - \int_0^1 r^3\,dr\right)d\varphi =$$

$$\begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix}=r$$

$$=\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 d\varphi = \int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{4}\right)d\varphi = \frac{1}{4}\int_0^{2\pi} d\varphi = \frac{2\pi}{4}=\frac{\pi}{2}\ \checkmark$$

N

Izračunaj integral $\iint_S \sqrt{\sqrt{x}+\sqrt{y}}\,dx\,dy$ za

$S=\{(x,y)\in\mathbb{R}^2;\ \sqrt{x}+\sqrt{y}\le 1,\ x,y\ge 0\}$

N

Let S območje, omejeno s krivuljami:

$y^2=px$, $y^2=qx$, $x^2=ay$, $x^2=by$ za

$y=\sqrt{p}\sqrt{x}$ $x=\sqrt{q}\sqrt{x}$ $y=\frac{1}{a}x^2$ $y=\frac{1}{b}x^2$

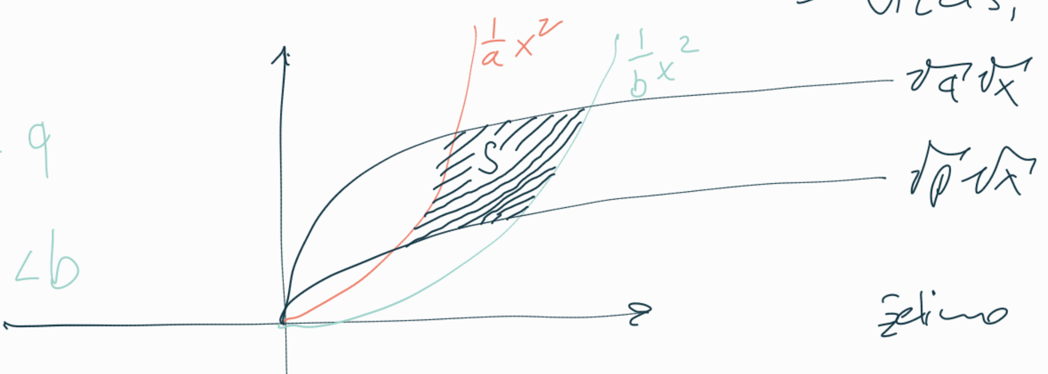
$0<p<q$ in $0<a<b$. Izračunaj

„lepo območje je stokrat vedno boljše kot lep integrand"
— vitas, 2024

$$\iint_S (x^3+y^3)\,dx\,dy =$$

$p<\frac{y^2}{x}<q$

$a<\frac{x^2}{y}<b$



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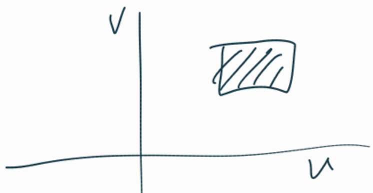


$$=\iint_\Omega \underbrace{uv(u+v)}_{\text{fja}}\ \underbrace{1/3}_{\text{det. Jacobi}}\,du\,dv$$

$u = \frac{x^2}{y} \qquad v = y^2/x \qquad \Rightarrow p < v < q$

$a < u < b$

mora biti bijekcija, torej

$\Omega = [a,b]_u \times [p,q]_v$

$x = \frac{y^2}{v} \Rightarrow u = \frac{\left(\frac{y^2}{v}\right)^2}{y} = \frac{y^4}{yv^2} = \frac{y^3}{v^2}$

$\Rightarrow y^3 = uv^2 \Rightarrow y = \sqrt[3]{uv^2}$

$x = \sqrt[3]{u^2 v}$ (zaradi simetrije)

$x^3 + y^3 = u^2v + uv^2 = uv(u+v)$

$$J = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3}(u^2v)^{-2/3}\,2uv & \frac{1}{3}(u^2v)^{-2/3}\,u^2 \\ \frac{1}{3}(uv^2)^{-2/3}\,v^2 & \frac{1}{3}(uv^2)^{-2/3}\,2uv \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{2}{3}u^{-1/3}v^{2/3} & \frac{1}{3}u^{2/3}v^{-2/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{vmatrix} = \frac{4}{9}u^0v^0 - \frac{1}{9}u^0v^0 = \frac{1}{3}$$

$$\overset{\text{fub.}}{\dots =} \frac{1}{3}\int_a^b\left(\int_p^q uv(u+v)\,du\right)dv = \frac{1}{3}\int_a^b u\int_p^q v(u+v)\,du\,dv =$$

$$= \frac{1}{3}\int_a^b u\left(u\frac{v^2}{2} + \frac{v^3}{3}\right)\Big|_p^q dv = \frac{1}{3}\int_a^b u\left(u\left(\frac{q^2}{2} - \frac{p^2}{2}\right) + \left(\frac{q^3}{3} - \frac{p^3}{3}\right)\right)du =$$

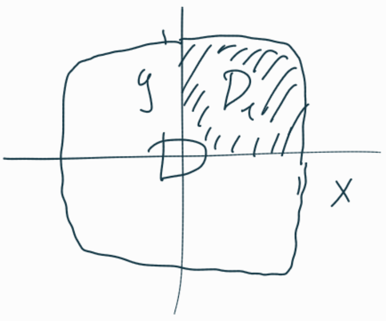
$$= \frac{1}{3}\left(\frac{u^3}{3}\left(\frac{q^2}{2} - \frac{p^2}{2}\right) + \frac{u^2}{2}\left(\frac{q^3}{3} - \frac{p^3}{3}\right)\right)\Big|_a^b$$

N

let $D = \{(x,y) \in \mathbb{R}^2 ; x^4 + y^4 \leq 1\}$ izračunaj

vsled simetrij integranda in območja

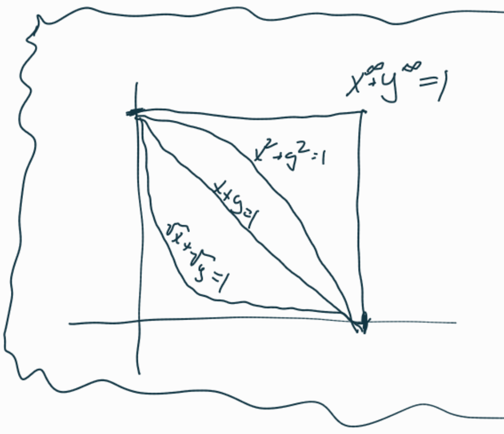
$$\iint_D (x^2+y^2)\,dx\,dy =$$



$$= 4\iint_{D_1}(x^2+y^2)\,dx\,dy = 4\iint(u+v)$$

$x = \sqrt{u} \quad y = \sqrt{v} \qquad \Omega = \{(x,y) \in \mathbb{R}^2 ; x,y \geq 0 ; x^2+y^2 \leq 1\}$

$$dx\,dy = J\,du\,dv = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$



$$\det J = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4\sqrt{uv}}$$

$$u = r\cos\varphi \qquad \det J = r$$
$$v = r\sin\varphi$$

$$\ldots = 4 \iint_{[0,1]_r \times [0,2\pi]_\varphi} (r\cos\varphi + r\sin\varphi) \frac{1}{4\sqrt{r\cos\varphi}\,\sqrt{r\sin\varphi}}\, r\,dr\,d\varphi =$$

$$= 4\int_0^1 r\,dr \cdot \int_0^{2\pi} \frac{\cos\varphi + \sin\varphi}{\sqrt{\cos\varphi}\,\sqrt{\sin\varphi}}\,d\varphi = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\varphi}}{\sqrt{\sin\varphi}}\,d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}}\varphi\,\sin^{-\frac{1}{2}}\varphi\,d\varphi =$$

$$= \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$\frac{1}{2} = 2q-1$

$-\frac{1}{2} = 2p-1$

$\frac{3}{2} = 2q \qquad q = \frac{3}{4}$

$\frac{1}{2} = 2p \qquad p = \frac{1}{4}$