ANA3VFMF 2024-10-28

kviz: naslednji teden
dn: do nedelje, danes objavljena

Utemelji, da integral $\int_0^\infty \frac{\ln(x^2+t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}}\,dt$ konvergira enakomerno za $x \in [0,1]$.

let $f(x,t) := \frac{\ln(x^2+t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}}$

$$\int_0^\infty f(x,t)\,dt = \int_0^{0{,}001} f(x,t)\,dt + \int_{0{,}001}^{1000} f(x,t)\,dt + \int_{1000}^\infty f(x,t)\,dt$$

zvezna f-ja na kompaktu enakom. konvergira (majoranta za njo bo max na D_f in ta enak. konv.)

Kritične točke:

$t = \infty$ (vedno): !

$t = 0$: !

$x=0, t=0$: !

$t<0$: ni v integracijskem intervalu ✓

V okolici $t \in (0,\infty)$ integral konvergira enakomerno, saj je integrand tam povsod definirana elementarna f-ja.

Weierstrassov M-test (majoranta): (za singularnost $t=\infty$)

$$\left|\frac{\ln(x^2+t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}}\right| \underset{x=1}{\leq} \frac{\ln(1^2+t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}} \lesssim \frac{\ln(1+t^2)}{(1+t^2)\sqrt{\pi/2}} \lesssim \frac{t^{1/2}}{\sqrt{\pi/2}\; t^2}$$

↳ za zelo velik t je pravzaprav konstanta

todaj: $\ln(1+t^2) \lesssim \sqrt[4]{1+t^2} \approx t^{1/2}$

ali je integral $\int_{1000}^\infty \frac{\ln(1+t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}}\,dt < \infty$?

$\int_{1000}^\infty \frac{t^{1/2}}{\sqrt{\pi/2}\; t^2}\,dt$? ja, ta pa konvergira

ŠE V $t=0$:

$$\left|\frac{\ln(x^2+t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}}\right| \underset{x=0}{\leq} \frac{|\ln(t^2)|}{(1+t^2)\sqrt{\operatorname{atan} t}} \lesssim \frac{2|\ln(t)|}{1\cdot\sqrt{t}} \leq \frac{2t^*}{\sqrt{t}} = \frac{2t^{-\frac14}}{\sqrt{t}} = \frac{2}{t^{3/4}}$$

$1+t^2 \approx 1$ in $\operatorname{atan} t \approx t$ za $t \ll 1$

$* = -\frac14$

↳ $|\ln(t)| < t^{-1/4}$ za $t \ll 1$

ali $\int_0^{0{,}001} \frac{\ln(t^2)}{(1+t^2)\sqrt{\operatorname{atan} t}}\,dt$ konvergira?

DA, saj $\int_0^{0{,}001} \frac{2}{t^{3/4}}\,dt < \infty$

N

Za $b>a>0$ izračunaj

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$$

ENA MOŽNOST: odvajamo po parametru a

DRUGA MOŽNOST: integrand prepoznamo kot nek drug integral:

$$\int_0^\infty\left(\int_a^b (\dots)\,dx\right)dt$$

$$e^{-xt}$$

$$\int_0^\infty\int_a^b e^{-xt}\,dx\,dt \;\boxed{\overset{*}{=}}\; \int_a^b\int_0^\infty e^{-xt}\,dt\,dx = \int_a^b \frac{e^{-xt}}{-x}\Bigg|_{t=0}^{t=\infty} dx =$$

$$=\int_a^b\left(\frac{0}{-x}-\frac{1}{-x}\right)dx = \int_a^b \frac{1}{x}\,dx = \ln|x|\Bigg|_{x=a}^{x=b} = \ln|b|-\ln|a|$$

$*$: (zamenjava vrstnega reda integriranja)

Vrsta Fubinijevega izreka, izposojena iz prihodnjih predavanj:
„vrstni red integriranja lahko zamenjamo, če je integrand (tule e^{-xt}) pozitiven".

N

Izračunaj integral

$$B(p,q)=2\int_0^{\pi/2}\sin^{2p-1}\varphi\,\cos^{2q-1}\varphi\,d\varphi$$

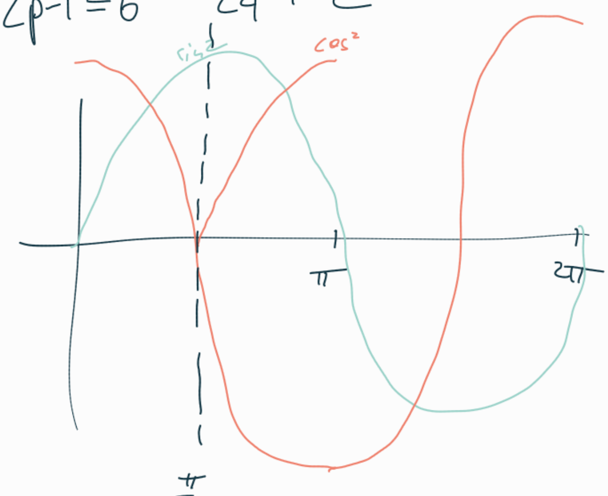
$$\int_0^{\pi}\sin^6 t\cos^2 t\,dt =$$

$$=B\left(\frac{7}{2},\frac{3}{2}\right)=$$

$$=\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}+\frac{3}{2}=5\right)}=$$

$$=\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{4!}=$$

$2p-1=6 \quad 2q-1=2$



$\frac{\pi}{2}$

simetrija!

$$= \frac{\frac{5}{2}\Gamma\left(\frac{5}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4!} = \frac{\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4!} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4!} =$$

$$= \frac{15\sqrt{\pi}^2}{16\cdot 4!} = \frac{\pi\cdot 15}{16\cdot 4!} = \frac{\pi\cdot 3}{16\cdot 4\cdot 2}$$

$$\left\{ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right.$$

(NTSBMPIL)

N

Za $q \in (0,1)$ izračunaj

$$\int_0^{\frac{\pi}{2}} \tan^q t \, dt = \int_0^{\frac{\pi}{2}} \frac{\sin^q t}{\cos^q t} dt = \int_0^{\frac{\pi}{2}} \sin^q t \cos^{-q} t \, dt =$$

$$= \frac{1}{2} B\left(\frac{q+1}{2}, \frac{1-q}{2}\right) =$$

$$= \frac{1}{2} \frac{\pi}{2\sin\left(\pi \cdot \frac{q+1}{2}\right)}$$

Eulerjeva refleksijska formula:

$$B(p, 1-p) = \frac{\pi}{\sin(\pi p)}$$

za $p \in (0,1)$

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za $p \in (-1,1)$ izračunaj $\int_0^\infty \frac{t^p \ln t}{1+t^2} dt =$

$u = t^2 \quad t = \sqrt{u}$
$du = 2t\,dt$
$dt = \frac{du}{2t} = \frac{du}{2\sqrt{u}}$

$$= \int_0^\infty \frac{t^p \ln t}{1+u} \cdot \frac{du}{2\sqrt{u}} =$$

mal!

$$= \frac{1}{4} \int_0^\infty \frac{\sqrt{u}^p \ln \sqrt{u}}{(1+u)\sqrt{u}} du = \frac{1}{4} \int_0^\infty \frac{u^{\frac{p}{2}-\frac{1}{2}} \ln u^{\frac{1}{2}}}{(1+u)} du =$$

$$= \frac{1}{4} \frac{\partial}{\partial x} B(x, 1-x) \Big|_{x=\frac{p+1}{2}}^{x=\infty} =$$

$$= \frac{1}{4} \frac{\partial}{\partial x} \frac{\pi}{\sin(\pi x)} \Big|_{x=\frac{p+1}{2}} = \dots$$

$$\left\{ B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du \right.$$

$$B\left(\frac{p+1}{2}, \frac{1-p}{2}\right) = \int_0^\infty \frac{u^{\frac{p-1}{2}}}{1+u} du$$

$$x + 1 = \frac{p+1}{2}$$

$$B(x, 1-x) = \int_0^\infty \frac{u^{x-1}}{1+u} du$$

odvajaj po x

$$\frac{\partial}{\partial x} B(x, 1-x) \overset{*}{=} \int_0^\infty \frac{u^{x-1}\ln(u)}{1+u} du$$

DOMAČA NALOGA:

Izračunaj $\int_0^{\pi/2} \frac{t}{\sqrt{\tan t}} dt$

asistent pravi, da je treba vadit zadeve Γ in B fj