ANA3VFMF 2024-10-21

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1.) Izračunaj odvod f-je
$$F(x)=\int_0^{x^3}\frac{\ln(1+xt)}{t}\,dt$$

$$F(x)=\int_{a(x)}^{b(x)} f(x,t)\,dt$$
$$\Longrightarrow \quad F'(x)=\int_{a(x)}^{b(x)} f_x(x,t)\,dt+b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))$$

Pogoj: $a(x)$ in $b(x)$ odvedljivi in $f_x(x,t)$ ter $f(x,t)$ sta zvezni f-ji dveh spremenljivk.

$$F'(x)\overset{*}{=}\int_0^{x^3}\frac{1}{1+xt}\,dt+3\cancel{x^2}\,\frac{\ln(1+x^4)}{x^{\cancel{3}}}-0=$$

$$=\int_0^{x^3}\frac{1}{1+xt}\,dt+\frac{3\ln(1+x^4)}{x}=$$

$$=\frac{\ln(1+xt)}{x}\Bigg|_{t=0}^{t=x^3}+\frac{3\ln(1+x^4)}{x}=$$

$$=\frac{\ln(1+x^4)}{x}+\frac{3\ln(1+x^4)}{x}=\frac{4\ln(1+x^4)}{x}$$

* utemeljitev: $x\mapsto x^3$ je zvezna, $x\mapsto 0$ je zvezna,
$x,t\mapsto\frac{\ln(1+xt)}{t}$ je zvezna in odvedljiva ✓
$x,t\mapsto\frac{1}{1+xt}$ je zvezna.

težave: $t=0$, $\underline{1+xt\leq 0}$

točke nedefiniranosti

t je istega predznaka kot x, torej je problem le v $x=0$ ali $t=0$

$$\frac{\ln(1+xt)}{t} \approx \frac{x\cancel{t}}{\cancel{t}} = x \quad , \quad \frac{\ln(1+xt)}{t} \longrightarrow x \quad \text{ko} \quad t \to 0$$

↳ za male x je $\ln(1+x) \approx x$

$\Rightarrow$ limita pri $t=0$ in $x = x_0$ obstaja $\Rightarrow$ $D_F = \mathbb{R}$

N

izračunaj

$$\int_0^{\frac{\pi}{2}} \ln \sin t \, dt = ?$$

vpeljemo x:

$$\int_0^{\frac{\pi}{2}} \ln\left(\sin^2 t + x\cos^2 t\right) dt$$

pri $F(0)$:

$$F(0) = 2\int_0^{\frac{\pi}{2}} \ln \sin t \, dt$$

pri $F(1)$:

$$F(1) = \int_0^{\frac{\pi}{2}} \ln 1 \, dt = 0$$

$$x \in (0, 1]$$

$$F'(x) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial x} \ln\left(\sin^2 t + x\cos^2 t\right) dt =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 t \quad :\cos^2 t}{\sin^2 t + x\cos^2 t \quad :\cos^2 t} dt = \int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 t + x} dt =$$

$$= \frac{1}{x}\int_0^{\frac{\pi}{2}} \frac{1}{1 + \left(\frac{1}{\sqrt{x}}\right)^2 \tan^2 t} dt \quad \underset{\dots}{\text{uvede primerne vrednosti}} \quad \frac{1}{x} \cdot \frac{\pi}{2} \cdot \frac{1}{1 + \frac{1}{\sqrt{x}}}$$

$$F(x) = \int \frac{1}{x} \cdot \frac{\pi}{2} \cdot \frac{1}{1 + 1/\sqrt{x}} dx = \frac{\pi}{2}\int \frac{1}{1+u} \cdot \left(-2u^{-1}\right) du =$$

$$= -\pi \int \frac{1}{u(1+u)} du =$$

$$= -\pi \int \left(\frac{A}{u} + \frac{B}{1+u}\right) du =$$

$u = x^{-\frac{1}{2}}$

$du = -\frac{1}{2} x^{-\frac{3}{2}} dx$

$du \cdot x^{1/2} = -\frac{1}{2} \, 1/x \, dx \qquad -2\, du \cdot u^{-1} = \frac{1}{x} dx$

$$\frac{A(1+u)+Bu}{u(1+u)}=\frac{1}{u(1+u)}$$

$$A+Au+Bu=1$$

$$A=1;\ B=-1$$

$$=-\pi\int\left(\frac{1}{u}-\frac{1}{1+u}\right)du=$$

$$=-\pi\left(\int\frac{1}{u}du-\int\frac{1}{1+u}du\right)=$$

$$=-\pi\left(\ln u-\ln(1+u)\right)+C=$$

$$=\pi\ln\frac{1+u}{u}+C=$$

$$=\pi\ln\frac{1+x^{-1/2}}{x^{-1/2}}+C=$$

$$=\pi\ln\left(1+\sqrt{x}\right)+C=F(x)=$$

iščemo še konstanto $C$.

$F(1)=0$ vemo.

$$\pi\ln(1+\sqrt{1})+C=0$$

$$\pi\ln 2=-C$$

$$C=-\pi\ln 2$$

$$=F(x)=\pi\ln\left(1+\sqrt{x}\right)-\pi\ln 2$$

za $x\in(0,1]$.

$$F(0)=\int_0^{\frac{\pi}{2}}\ln\left(\sin^2 t+0\cdot\cos^2 t\right)dt=$$

$$=2\int_0^{\frac{\pi}{2}}\ln\sin t\,dt$$

$$F(0)\overset{**}{=}\lim_{x\downarrow 0}F(x)=\lim_{x\downarrow 0}\pi\ln\left(1+\sqrt{x}\right)-\pi\ln 2=-\pi\ln 2,\ \checkmark$$

torej $\int_0^{\frac{\pi}{2}}\ln\sin t\,dt=\frac{-\pi\ln 2}{2}$

utemeljitev $**$: zveznost $F$ v $[0,1]$:

pogoji: • $f(x,t)=\ln(\sin^2 t+x\cos^2 t)$ zvezna?

ja, ker je elementarna

na $[0,1]_x\times[0,\frac{\pi}{2}]_t$

problem v $x=0,\ t=0$:

$\ln\left(\sin^2 t+x\cos^2 t\right)\longrightarrow-\infty$ (RE,

$f$ ni omejena $\Rightarrow$ ni zvezna

HM...

Če uspemo pokazati, da je $\int_0^{\pi/2} \ln(\sin^2 t + x\cos^2 t)$ zvezna f-ja za $x \in [0,1]$, velja

$$\int_0^{\pi/2} \ln \sin t \, dt = \frac{-\pi \ln 2}{2}$$

. problem: $f(x,t)$ ni zvezna!

Izrek: (to izrek 1 ne dela oz. imamo nestonične meje). $a=\infty, b=\infty$, ali pa $f$ ni omejena

$$F(x) = \int_a^b f(x,t)\, dt ,$$

$F$ je zvezna na $[c,d]_x$, če je $f$ zvezna in

$F(x) = \int_a^b f(x,t)\, dt$ konvergira enakomerno:

Weierstrassov M-test:

če $|f(x,t)| \leq g(t)$ ← neodvisna od $x$

in $\int_a^b g(t)\, dt < \infty \Rightarrow \int_a^b f(x,t)\, dt$ konvergira enakomerno.

Da utemeljimo zveznost $F$, moramo pokazati, da $F$ konvergira enakomerno v okolici $x=0$.

$$|f(x,t)| \leq g(t) = \sup_{x \in} f(x,t)$$

$$\parallel$$

$$|\ln(\sin^2 t + x\cos^2 t)|$$

iščemo supremum. največ (upoštevaje abs), ko je to minimalno, kar se zgodi pri najmanjšem $x$, torej

kada $x \geq 0$

$x=0$: $\quad -\ln \sin^2 t = g(t)$

$$|\ln(\sin^2 t + x\cos^2 t)| \leq |\ln \sin^2 t| .$$

ali obstaja $\int_0^{\frac{\pi}{2}} |\ln \sin^2 t| \, dt < \infty$ ? potencialen problem je pri $t=0$ (neskončnost), kjer je integrand omejen.

majhen $t$

$$|\ln \sin^2 t| \approx 2\,|\ln t| \lesssim 2\, t^{-\frac{1}{2}} = 2\,\frac{1}{\sqrt{t}}$$

$$\int_0^{\frac{\pi}{2}} |\ln \sin^2 t|\,dt \lesssim \int_0^{\frac{\pi}{2}} 2\,\frac{1}{\sqrt{t}}\,dt$$ za male $t$ v okolici 0

„logaritem je konstanta', ki raste skozi čas"
-- en profesor fizike.

$\ln t$ za $t \gg 1$ gre počasi proti $\infty$.

$\ln t$ za $t \ll 1$ gre počasi proti $-\infty$.

velja za $t \gg t$:

$$1 < \ln t < t^{\alpha} \quad \text{za } \alpha > 0$$

podobno velja za $t \ll 1$:

$1 < |\ln|t|| < t^{-\alpha}$, $\alpha > 0$: (počasno padanje proti $-\infty$)

$$\lim_{t\downarrow 0} \frac{\ln t}{t^{-\alpha}} \overset{L'H.}{=} \lim_{t\downarrow 0} \frac{\frac{1}{t}}{-\alpha t^{-\alpha-1}} = \lim_{t\downarrow 0} \frac{1}{-\alpha t^{-\alpha}} = 0, \text{ torej}$$

gre $t^{-\alpha}$ hitreje proti $-\infty$ kot $\ln t$.

utemeljili smo absolutno konvergenco $\Rightarrow$ $F$ zvezna.

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Izračunaj

$$\int_0^{\infty} \frac{e^{-t}}{t} \sin t\,dt = ?$$

$$F(x) = \int_0^{\infty} \frac{e^{-xt}}{t} \sin t\,dt$$

$x \in (0, \infty)$

pri $x = 1$:

$$F(1) = \int_0^{\infty} \frac{e^{-t}}{t} \sin t\,dt$$

pri $x = ?$ $F(?) = \dots$ znamo izračunati.

$$F'(x) \overset{*}{=} \int_0^\infty \frac{-\cancel{t}\, e^{-xt}}{\cancel{t}} \sin t \, dt = -\int_0^\infty \frac{e^{-xt}}{t} \sin t \, dt =$$

(**)

$u = e^{-xt}$
$du = -xe^{-xt}$
$dv = \sin t$
$v = -\cos t$

$$= -\left(e^{-xt}(-\cos t)\right)\Big|_0^\infty + \int \cos t \cdot (-x e^{-xt}) \, dt =$$

$$= 0 - 1 + x \int_0^\infty \cos t \cdot e^{-xt} dt = -1 + x \int_0^\infty \cos t \cdot e^{-xt} =$$

$u = e^{-xt}$
$du = -xe^{-xt}$
$dv = \cos t$
$v = \sin t$

$$= -1 + x\left(e^{-xt} + \sin t\right)\Big|_0^\infty - x \int_0^\infty \sin t \cdot xe^{-xt} dt =$$

$$= -1 + 0 + x^2 \underbrace{\int_0^\infty \sin t \cdot e^{-xt} dt}_{-F'(x)}$$

$$F'(x) = -1 + x^2(-F'(x))$$

$$(1+x^2) F'(x) = -1$$

$$F'(x) = -\frac{1}{1+x^2} \Rightarrow F(x) = -\int \frac{1}{1+x^2} dx = -\operatorname{atan} x + C$$

$C = ?$

Znamo izračunati $F(\infty) = 0$.

$$\lim_{x\to\infty}(\operatorname{atan} x + C) = -\frac{\pi}{2} + C$$

$$C = \frac{\pi}{2}$$

$$\int_0^\infty \frac{e^{-t}}{t} \sin t \, dt \overset{x=1}{=} -\operatorname{atan} 1 + \frac{\pi}{2} = \frac{\pi}{4}$$

utemeljitev *: problem v $t = \infty$

$$F(x) = \int_a^{\infty} f(x,t)\,dt$$

$$F'(x) = \int_a^\infty \frac{\partial}{\partial x} f(x,t)\,dt, \text{ pogoj:}$$

če sta $f$ in $f_x$ zvezni in

$\int_a^\infty f_x(t)\,dt$ konvergira enakomerno.

Weierstraßov M-test:

$$|f_x(x,t)| \le g(t)$$

uaf m. v $x = 1$

$$|-e^{-xt}\sin t| \le g(t) = \sup_x -e^{-xt} \sin t = e^{-t} \sin t$$

konvergira?

$$\int_0^\infty e^{-t} \sin t < \infty \quad \text{DA.} \quad (**)$$

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