ANA3VFMF2024-10-14

izračunaj

$$\int_0^{\pi/2} \frac{t}{\tan t}\,dt$$ — za $\int \frac{t}{\tan t}\,dt$. primitivna fja ni elementarna.

Definirajmo integral s parametrom:

$$F(x) = \int_0^{\pi/2} \frac{\operatorname{atan}(x \tan t)}{\tan t}\,dt$$

(*) definirajmo, da F ni definirana v $x = \pm 1$

$$F'(x) \overset{(**)}{=} \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial x} \frac{\operatorname{atan}(x \tan t)}{\tan t}\,dt = \int_0^{\frac{\pi}{2}} \frac{1}{\tan t} \frac{1}{1+x^2\tan^2 t} \cdot \tan t\,dt =$$

$$= \int_0^{\pi/2} \frac{1}{1+x^2\tan^2 t}\,dt = \int_0^{\infty} \frac{1}{1+(xu)^2} \cdot \frac{1}{1+u^2}\,du =$$

$u = \tan t$

$t = \operatorname{atan} u$

$dt = \frac{1}{1+u^2}\,du$

$t=0 \Rightarrow u=0$

$t=\frac{\pi}{2} \Rightarrow u = \tan\frac{\pi}{2} = \infty$

$$= \int_0^{\infty} \frac{Au+B}{1+x^2u^2} + \frac{Cu+D}{1+u^2}\,du = \dots (*)$$

$$= \int_0^{\infty} \frac{(Au+B)(1+u^2) + (Cu+D)(1+x^2u^2)}{(1+x^2u^2)(1+u^2)}\,du =$$

$$= \int_0^{\infty} \frac{Au + Au^3 + B + Bu^2 + Cu + Cx^2u^3 + D + Dx^2u^2}{1+u^2+x^2u^2+x^2u^4}\,du =$$

$$\frac{1}{\cancel{1+u^2+x^2u^2+x^2u^4}} = \frac{Au+Au^3+B+Bu^2+Cu+Cx^2u^3+D+Dx^2u^2}{\cancel{1+u^2+x^2u^2+x^2u^4}}$$

$$1 = Au + Au^3 + B + Bu^2 + Cu + Cx^2u^3 + D + Dx^2u^2$$

$$1 = u^3(A + Cx^2) + u^2(B + Dx^2) + u(A + C) + 1(B + D)$$

$$0 = A + Cx^2 \qquad A = -Cx^2 \qquad \Rightarrow A = 0$$

$$0 = B + Dx^2 \qquad B = -Dx^2$$

$$0 = A + C \qquad -Cx^2 + C = 0 \Rightarrow C(-x^2 + 1) = 0 \Rightarrow$$

$$1 = B + D \qquad C = 0 \text{ za } x \neq \pm 1 \quad (*)$$

$$1 = -Dx^2 + D = D(-x^2 + 1)$$

$$D = \frac{1}{1 - x^2}$$

$$A = 0, \quad B = \frac{x^2}{x^2 - 1}, \quad C = 0, \quad D = \frac{1}{1 - x^2}$$

$$B = -Dx^2 = \frac{-x^2}{1 - x^2} = \frac{x^2}{x^2 - 1}$$

$$(*) \ldots = \int_0^{\frac{\pi}{2}} \frac{Au + B}{1 + x^2u^2} + \frac{Cu + D}{1 + u^2} = \int_0^{\frac{\pi}{2}} \frac{x^2}{(x^2 - 1)(1 + x^2u^2)} + \frac{1}{(1 - x^2)(1 + u^2)}\, du =$$

$$= \ldots = \frac{-x^2}{1 - x^2} \cdot \frac{1}{x} \cdot \operatorname{atan}(xu) + \frac{1}{1 - x^2} \operatorname{atan} u \Big|_0^{\infty} =$$

$$= \frac{-x}{1 - x^2} \cdot \frac{\pi}{2} + \frac{\pi/2}{1 - x^2} - 0 = \frac{\pi}{2}\left(\frac{-x + 1}{1 - x^2}\right) = \frac{\pi}{2} \frac{1 - x}{(1 - x)(1 + x)} = \frac{\pi}{2} \cdot \frac{1}{1 + x}$$

Izračunali smo, da je $F'(x) = \frac{\pi}{2} \cdot \frac{1}{1 + x}$ za $x \in [0, \infty) \setminus \{1\}$.
velja tudi za $x = 1$ zaradi zveznosti integranda $\frac{1}{1 + (x \tan t)^2}$ v definiciji F'.

$$F(x) = \int \frac{\pi}{2} \frac{1}{1 + x}\, dx = \frac{\pi}{2} \int \frac{1}{1 + x}\, dx = \frac{\pi}{2} \ln(1 + x) + C$$

Koliko je C? oglejmo si definicijo F. očitno je $F(0) = 0$,
torej $\frac{\pi}{2} \ln(1 + 0) + C = 0 \Rightarrow \frac{\pi}{2} \cdot 0 + C = 0 \Rightarrow \underline{C = 0}$

prvotni integral: $\displaystyle\int_0^{\frac{\pi}{2}} \frac{t}{\tan t}\,dt = F(1) = \frac{\pi}{2}\ln(1+1) = \frac{\pi}{2}\ln 2$

(**) zakaj to smemo početi? ker je integrand zvezen na $[0,\frac{\pi}{2}] \times \mathbb{R}^+ \setminus \{1\}$

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izračunaj $\displaystyle\int_0^1 \frac{1-t}{\ln t}\,dt$

definirajmo: $\displaystyle F(x) = \int_0^1 \frac{t^x - 1}{\ln t}\,dt$ na intervalu $(-1, \infty)$.

vsebuje 0, da $F(0) = 0$

vsebuje 1, da $\displaystyle F(1) = \int \frac{1-t}{\ln t}\,dt$

$$F'(x) \overset{(**)}{=} \int_0^1 \frac{\partial}{\partial x}\frac{t^x - 1}{\ln t}\,dt = \int_0^1 \frac{1}{\ln t} \ln t \cdot t^x\,dt = \int_0^1 t^x\,dt =$$

$$= \frac{t^{x+1}}{x+1}\Big|_0^1 = \frac{1^{x+1}}{x+1} - 0 = \frac{1}{x+1}$$

$$F(x) = \int F'(x)\,\partial x = \int \frac{1}{x+1}\,\partial x = \ln(x+1) + C$$

določi C: $\quad F(0) = 0 = \ln(1) + C \Rightarrow C = 0$

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utemeljimo (**): a.) $\frac{t^x - 1}{\ln t}$ je zvezna na $(-1, \infty)_x \times [0,1]_t$

b.) t^x je zvezna na istem

a) elementarna f-ja, problem v $t=0$ in $t=1$

$$\lim_{(x,t)\downarrow(x_0,0)} \frac{t^x-1}{\ln t} = 0$$

($\ln t \to -\infty$)

$$\lim_{(x,t)\uparrow(x_0,1)} \frac{t^x-1}{\ln t} = \lim_{(x,t)\to(x_0,1)} \left(\frac{t^x-1}{x\ln t}\right) x = \lim_{(x,t)\to(x_0,1)} \frac{t^x-1}{\ln t^x}\cdot x =$$

$$= \lim_{t^x\to 1} \frac{t^x-1}{\ln t^x} \lim_{x\to x_0} x_0 = x_0 \lim_{u\to 1} \frac{u-1}{\ln u} \overset{L.H.}{=} x_0 \lim_{u\to 1} \frac{1}{1/u} = x_0$$

b) t^x je zvezna na istem (je elementarna)

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So 8E02

Izračunaj $\int t^2 e^{-t}\, dt$

Def.: $F(x) = \int_*^* t^2 e^{-xt}\, dt$

$G(x) = \int_*^* e^{-xt}\, dt$

$G''(x) = F(x)$

...

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Izračunaj $\lim_{x\to 0} \int_x^{1-x} \frac{e^{xt}}{1+t^2+x^2}\, dt \overset{(**)}{=} \int_0^{1-0} \frac{e^{0t}}{1+t^2+0}\, dt =$

$$= \int_0^1 \frac{1}{1+t^2}\, dt = \operatorname{atan} 1 - \operatorname{atan} 0 = \frac{\pi}{4}$$

(**) Izrek: f-ja $F(x) = \int_{a(x)}^{b(x)} f(x,t)\, dt$ je zvezna, če:

$a(x)$ je zvezna, $b(x)$ je zvezna in $f(x,t)$ je zvezna na $\{(x,t)\in\mathbb{R}^2 ;\ a(x)\le t\le b(x)\}$ $= [c,d]_x \times [a(x), b(x)]_t$

a in b sta očitno zvezni, $\frac{e^{xt}}{1+t^2+x^2}$ je zvezna, ker je element.

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$$F(x) = \int_0^{x^3} \frac{\ln(1+xt)}{t}\, dt.$$ izračunaj F'

$$\frac{d}{dx}\int_0^x f(t)\, dt \overset{\text{o.i.a.}}{=} f(x)$$

$$\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{d}{dx} f(x,t)\, dt$$

sestavimo.

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\, dt$$

$$G(x,a,b) = \int_a^b f(x,t)\, dt$$

$$F(x) = G(x, a(x), b(x))$$

$$\frac{d}{dx} F(x) = G_x x' + G_a a' + G_b b'$$

KOLOKVIJ:

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- v januarju začnemo ob 9h