ANA3 PFMF 2024-11-05

Fubinijev izrek: če je f zvezna in integrabilna na $[a,b]\times[c,d]$ in je $x \mapsto f(x,y)$ integrabilna na $[a,b]$ $\forall y \in [c,d]$ in $y \mapsto f(x,y)$ integrabilna na $[c,d]$ $\forall x \in [a,b]$,

potem
$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

Dokaz Fubinijevega izreka:

Let $g(x) = \int_c^d f(x,y)\,dy$ za $x \in [a,b]$ (integral s parametrom).

dokazati moramo, da
$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b g(x)\,dx.$$

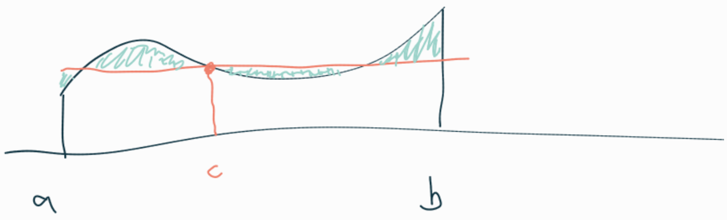
Riemannova vsota za desni integral je $R(g, D_x, \vec{t}) =$
$$= \sum_{j=1}^n g(\underbrace{t_j}_{\text{testne t.}})\underbrace{(x_j - x_{j-1})}_{\text{delilne točke}} = \sum_{j=1}^n \int_c^d f(t_j,y)\,dy \cdot (x_j - x_{j-1})$$

Ključna ideja bo integral $[c,d]$ razbiti na manjše kose, ti integrirajo po $y_0,\dots,y_m$ (delitev $[c,d]$: $c = y_0 < y_1 < \dots < y_m = d$). Delitev označimo z D_y.

$$\int_c^d f(t_j,y)\,dy = \sum_{k=1}^m \int_{y_{k-1}}^{y_k} f(t_j,y)\,dy.$$

Iz ANA 1 poznamo izrek o srednji vrednosti:
pove, da $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ za nek $c \in [a,b]$

t.j. povprečje je doseženo v neki točki.



To uporabimo za naše integrale: $\exists\, s_k \in [y_{k-1}, y_k]$:
$$f(t_j, s_k) = \frac{1}{y_k - y_{k-1}}\int_{y_{k-1}}^{y_k} f(t_j,y)\,dy$$

$$\Rightarrow \int_{y_{k-1}}^{y_k} f(t_j,y)\,dy = f(t_j,s_k)(y_k - y_{k-1})$$

Sledi: $R(g, D_x, \tilde{t}) = \sum_{j=1}^{n}\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} f(t_j, y)\,dy\,(x_j - x_{j-1}) =$

$$= \sum_{\substack{j=1\\k=1}}^{\substack{n\\m}} f(t_j, s_k)(y_k - y_{k-1})(x_j - x_{j-1}) =$$

$\{s_1, \dots, s_k\}$

$$= R(f, D_x \times D_y, \tilde{t} \times \tilde{s}) =$$

Riemannova vsota fje z spr. f pri delitvi tč. (x_j, y_k) in pri vetem naboru testnih točk (t_j, s_k).

Vemo: $s(f, D_x \times D_y) \le R(f, D_x \times D_y, \tilde{t} \times \tilde{s}) \le S(f, D_x \times D_y)$

ker je f integrabilna, se v limiti te tri vrednosti ujamejo. $\square$

Primer uporabe: integrirajmo $f(x,y) = x + y^2$ na $[0,1]\times[0,1]$!

$$\iint_{[0,1]\times[0,1]} x + y^2 \, dx\,dy = \int_0^1 \left(\int_0^1 x + y^2 \, dy \right) dx = \int_0^1 xy + \frac{y^3}{3}\Big|_0^1 \, dx =$$

$$= \int_0^1 x + \frac{1}{3}\, dx = \frac{x^2}{2} + \frac{1}{3}x \Big|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

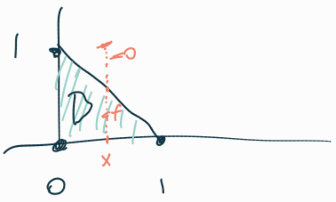
Sedaj obrnimo vrstni red:

$$\iint_{[0,1]\times[0,1]} x + y^2 \, dx\,dy = \int_0^1 \left(\int_0^1 x + y^2 \, dx \right) dy = \int_0^1 \frac{x^2}{2} + xy^2 \Big|_0^1 \, dy =$$

$$= \int_0^1 \frac{1}{2} + y^2 \, dy = \frac{1}{2}y + \frac{y^3}{3}\Big|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Kaj pa, če nismo na pravokotniku?

recimo na 3kotniku z oglišči $(0,0)$, $(1,0)$, $(0,1)$.



Po definiciji vzamemo razširitev $\tilde{f}(x) = \begin{cases} f(x); & x \in D \\ 0; & \text{sicer} \end{cases}$, definirano na $[0,1]\times[0,1]$, in jo integriramo na $[0,1]\times[0,1]$.

$$\iint_D f(x,y)\,dx\,dy \overset{\text{po definiciji}}{=} \iint_{[0,1]\times[0,1]} \tilde{f}(x,y)\,dx\,dy \overset{\text{fubini}}{=} \int_0^1 \left(\int_0^1 \underbrace{\tilde{f}(x,y)}\, dy \right) dx =$$

ta fja je enaka $f(x,y)$ za $y \in [0, 1-x]$
sicer je ničelna za $y \in [1-x, 1]$.

$$= \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx = \int_0^1 \left(\int_0^{1-x} x+y^2\,dy \right) dx = \int_0^1 xy + \frac{y^3}{3}\Big|_0^{1-x} dx =$$

$$= \int_0^1 x(1-x) + \frac{(1-x)^3}{3}\,dx = \int_0^1 x - x^2 + \frac{1}{3}\left(1-3x+3x^2-x^3\right) dx =$$

$$= \frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{3}x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{12}\Big|_0^1 =$$

$$= \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

sedaj obrnimo vrstni red!

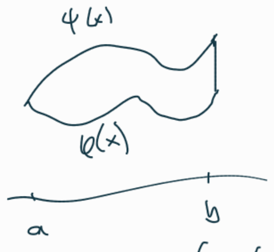
$$\iint_D f(x,y)\,dx\,dy = \iint_{[0,1]\times[0,1]} \tilde f(x,y)\,dx\,dy \overset{fub}{=} \int_0^1 \left(\underline{\int_0^1 \tilde f(x,y)\,dx} \right) dy =$$

nenič. le za $x \in [0, 1-y]$

$$= \int_0^1 \int_0^{1-y} x+y^2\,dx\,dy = \dots = \frac{1}{4} \quad \checkmark$$

Povzetek: kaj velja v splošnem?

če je $D = \{ (x,y) \in \mathbb{R} \,;\; x \in [a,b] \,;\; y \in [\varphi(x), \psi(x)] \}$



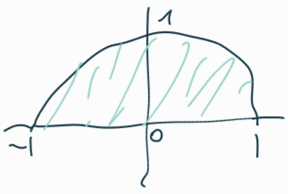
potem

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right) dx$$

KOMENTAR:

1. Iz primera na 3Δniku se morda zdi, da so meje po zamenjavi vrstnega reda „simetrične". Temu seveda ni tako (fcking obvious).

Primer:

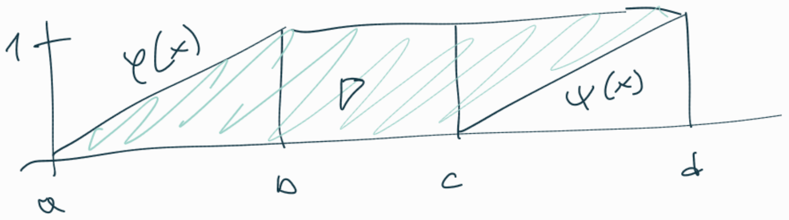


D je polovica enotskega kroga.

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

META
12.11.
19.11. 9h v P.01
Predavanja

2. včasih moramo območja razrezati tudi na več kosov



$$\iint_D f(x,y)\,dx\,dy = \int_0^a \left(\int_0^{\varphi(x)} f(x,y)\,dy\right)dx + \int_a^b \left(\int_0^1 f(x,y)\,dy\right)dx$$

$$+ \int_b^c \left(\int_{\psi(x)}^1 f(x,y)\,dy\right) dx$$

3. Doslej smo z oklepaji poudarjali vrstni red integracije. tega sedaj ne bomo več počeli. Uporabljamo notacijo:

$\iint_D f(x,y)\,dx\,dy$: dvojni integral f po D

$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b dx \int_c^d f(x,y)\,dy$: integral v izbranem vrstnem redu

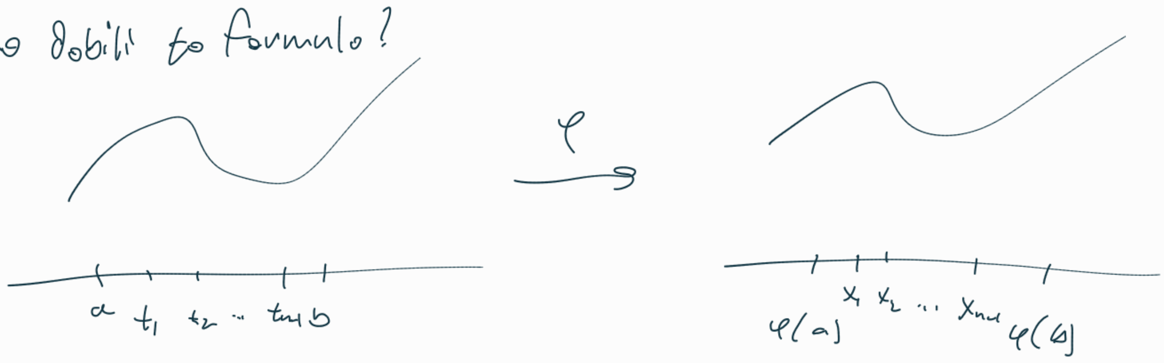
[uvedba novih koordinat v dvojni integral]

spomnimo se, kako to storimo pri fjah ene spremenljivke:

$$\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx = \int_a^b f(\varphi(t))\cdot \varphi'(t)\,dt$$

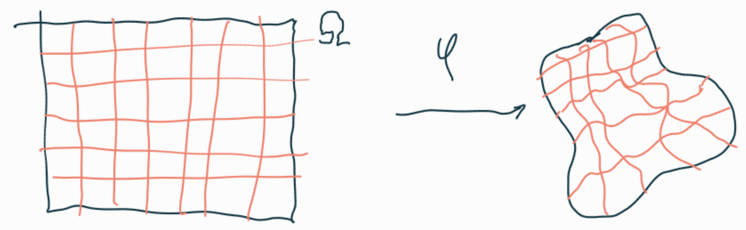
to formulo dobimo pri substituciji $x=\varphi(t)$, kjer je $\varphi:[a,b]\to\mathbb{R}$ zvezno odvedljiva a ne nujno bijektivna.

Kako smo dobili to formulo?



imamo ekvivalentne riemannove vsote, le širina intervalov v delitvi se spreminja, zato v končnem zapisu dodamo $dx=\varphi'(t)\,dt$

Analogno to storimo za fje več spremenljivk.



$\varphi(\Omega) = D$

pravokotnik $[x_{j-1}, x_j] \times [y_{k-1}, y_k]$ se preslika v krivočrtni analog

Ploščina tega pravokotnika je $|\det J| = \begin{vmatrix} x_t & x_s \\ y_t & y_s \end{vmatrix}$ na infinitezimalni ravni.

$\varphi(t,s) = \big(x(t,s), y(t,s)\big)$, kjer sta (t,s) spremenljivki na Ω, (x,y) pa na D

Posledica:

Izrek: let $\Omega \subset \mathbb{R}^2$ območje in $\varphi : \Omega \to \mathbb{R}^2$ preslikava z zvezno odv. komponentami. let $\det J_\varphi \neq 0$ na Ω in let $f : \varphi(\Omega) \to \mathbb{R}^2$ zvezna. potem velja:

$$\iint_{\varphi(\Omega)} f(x,y)\,dx\,dy = \iint_{\Omega} f(\varphi(t,s)) \cdot |\det J_\varphi(t,s)|\,dt\,ds$$

tako kot smo pri integralu fje ene spremenljivke dodali $\varphi'(t)$, moramo pri dveh dodati det Jacobijeve matrike.

Kaj je Jacobijeva matrika?

$f : \mathbb{R}^n \to \mathbb{R}^m$

$f(x_1, \dots, x_n) = \big(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\big)$

$$J_{f(x_1,\dots,x_n)} = \underbrace{\begin{bmatrix} (f_1)_{x_1} & \dots & (f_1)_{x_n} \\ \vdots & & \vdots \\ (f_m)_{x_1} & \dots & (f_m)_{x_n} \end{bmatrix}}$$

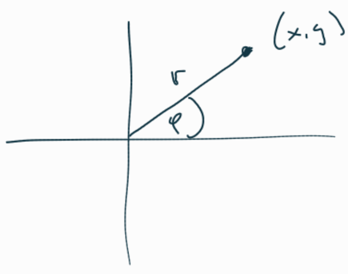
linearni približek

[uvedba novih koordinat v dvojni integral]

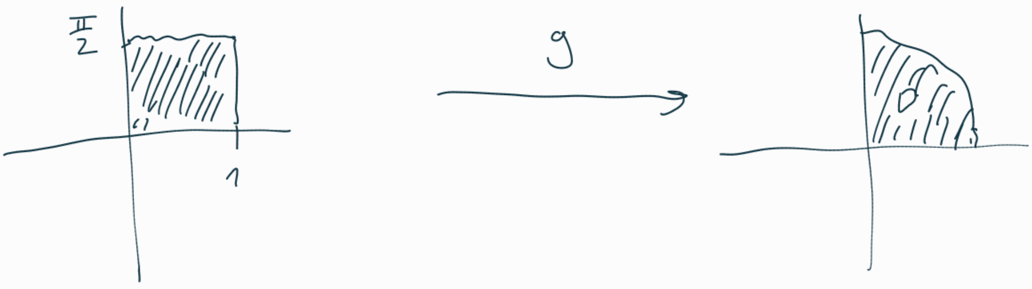
Zgled: polarne koordinate:

$x = r\cos\varphi$ $\quad r \geq 0$: polarni radij

$y = r\sin\varphi$ $\quad \varphi \in [0, 2\pi]$ polarni kot



recimo, da želimo integrirati fjo $f(x,y) = x$ na četrtini enotskega kroga v prvem kvadrantu.



$$g(r,\varphi) = (x(r,\varphi), y(r,\varphi)) = (r\cos\varphi, r\sin\varphi)$$

$$J_{g(r,\varphi)} = \begin{bmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

$$|\det J_{g(r,\varphi)}| = r\cos^2\varphi + r\sin^2\varphi = |r| = r$$

izrek

x v polarnih koordinatah

det Jacobija

$$\iint_D \boxed{x}\,dx\,dy = \iint_{[0,1]\times[0,\frac{\pi}{2}]} \boxed{r\cos\varphi} \cdot r \, dr\, d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^2 \cos\varphi \, dr = \int_0^{\frac{\pi}{2}} \frac{r^3}{3} \cos\varphi \Big|_0^1 d\varphi = \int_0^{\frac{\pi}{2}} \frac{1}{3}\cos\varphi\, d\varphi =$$

$$= \frac{1}{3}\sin\varphi \Big|_0^{\frac{\pi}{2}} = \frac{1}{3}$$

naprarimo preizkus v kartezičnih koordinatah.

$$\iint_D x\,dx\,dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} x\,dy = \int_0^1 xy\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 x\sqrt{1-x^2}\,dx =$$

$t = 1-x^2$
$dt = -2x\,dx$

$$= -\frac{1}{2}\int_1^0 \sqrt{t}\,dt = \frac{1}{2}\int_0^1 \sqrt{t}\,dt = \frac{1}{2}\frac{t^{3/2}}{3/2}\Big|_0^1 = 1/3 \quad \checkmark$$