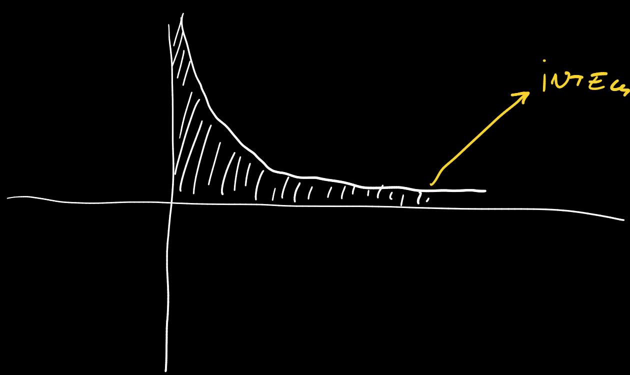
ANA3PFMF 2024-10-22

[GAMA IN BETA FUNKCIJI]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$ funkcija gama

Ta f-ja konvergira za poljuben $x \in \mathbb{R}$, saj eksponent e^{-t} pada hitreje kot t^{x-1}. Vendar pa pri $t=0$ $\exists$ le za $x>0$, saj sicer pol ni ustrezne stopnje.



Torej je $D_\Gamma = (0, \infty)$, premislimo pa tudi, da je Γ na njem zvezna f-ja.

IZREK: Γ je zvezna na $(0,\infty)$

DOKAZ: rabimo enakomerno konvergenco. Let $\varepsilon > 0$, $M>0$. dokažimo, da je Γ enak. konv. int. s parametrom na $[\varepsilon, M]$, posledično je tam zvezen, ker pa sta ε in M poljubna, to pove, da je Γ zvezna na $(0,\infty)$.

Potrebujemo le ustrezno majoranto:

$$\Gamma(x) = \int_0^1 \underline{t^{x-1} e^{-t}}\, dt + \int_1^\infty \underline{t^{x-1} e^{-t}}\, dt$$

v BLIŽINI $t=0$ se integrand obnaša kot t^{-1} oz. ga lahko omejimo s $t^{\varepsilon-1}$ za $x>\varepsilon$.

v bližini $t=\infty$ eksponent premaga t^{x-1} za poljuben x in lahko najdemo oceno, da je $t^{x-1}e^{-t} \le t^{M-1} e^{-t} \le t^{M-1} \frac{1}{t^{M+1}} = \frac{1}{t^2}$ za vse $x \le M$.

ker integrala $\int_0^1 t^{x-1}e^{-t}dt$ in $\int_1^\infty t^{x-1}e^{-t}dt$ konvergirata, tudi Γ konvergira enakomerno.

IZREK: $\forall x>0: \Gamma(x+1) = x\,\Gamma(x)$

DOKAZ: $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = -t^x e^{-t}\Big|_{t=0}^{t=\infty} + \int_0^\infty x t^{x-1} e^{-t}\,dt =$

PER PARTES:
$u = t^x$
$du = x t^{x-1}\,dt$
$dv = e^{-t}\,dt$
$v = -e^{-t}$

0, ker exp hitreje pada k 0 kot t^x raste proti ∞.

$= -\cancel{\lim_{t\to\infty} t^x e^{-t}} + \cancel{0 e^0} + x\,\Gamma(x) =$

$= 0 + x\,\Gamma(x)$ □

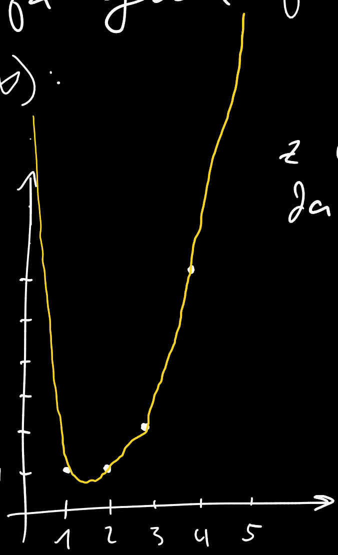
POSLEDICA: Če vzamemo naravna števila: $\forall n\in\mathbb{N}: \Gamma(n) = (n-1)!$

DOKAZ: $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)\cdots 1\cdot\Gamma(1) =$

$= (n-1)(n-2)\cdots 1\cdot \int_0^\infty t^0 e^{-t}\,dt = (n-1)(n-2)\cdots 1 \int_0^\infty e^{-t}\,dt =$

$= (n-1)(n-2)\cdots 1\cdot\left(-e^{-t}\Big|_{t=0}^{t=\infty}\right) = (n-1)(n-2)\cdots 1\cdot(-0+1) = (n-1)!$ □

SKLEP: fja gama je zvezna razširitev fakultete iz $\mathbb{N}$ na $(0,\infty)$:



z malce dodatne analize lahko preverimo, da $\Gamma'(x) = \left(\int_0^\infty t^{x-1}e^{-t}\,dt\right)' =$

$= \int_0^\infty \frac{\partial}{\partial x}\left(t^{x-1}e^{-t}\right)dt = \int_0^\infty t^{x-1}\ln t\, e^{-t}\,dt.$

Za se premisliti, da ima ničlo na $x\in(1,2)$!

$\Gamma''(x) = \int_0^\infty t^{x-1}\ln^2 t\, e^{-t}\,dt \geq 0,$

torej je fja konveksna.

[FUNKCIJA BETA]

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \quad ; \; p,q>0.$$

vidimo, da morata p in q nujno biti strogo pozitivna, saj sicer integrala ne obstaja pri $t=0$ ali $t=1$.

IZREK: $\forall p,q>0: \; B(p,q) = \dfrac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$.

POSLEDICA: B je zvezna f-ja 2 sprem. na $(0,\infty)\times(0,\infty)$ in da je simetrična (t.j. $B(p,q)=B(q,p)$) -- usled komutat.

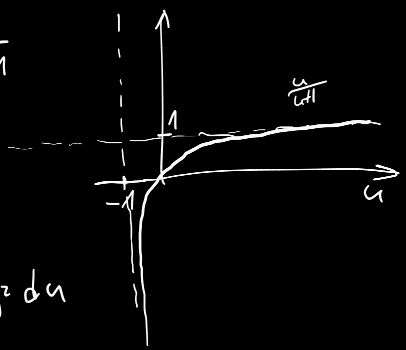
DOKAZ:

LEMA: $B(p,q) = \displaystyle\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\,du$

DOKAZ LEME: uvedi $t = \dfrac{u}{u+1}$

ko u teče od 0 do ∞,
t teče od 0 do 1

$$dt = \frac{u+1-u}{(u+1)^2}\,du = \frac{1}{(u+1)^2}\,du$$



ustavimo to substitucijo v B funkcijo:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \int_0^\infty \left(\frac{u}{u+1}\right)^{p-1}\Big(\underbrace{1-\frac{u}{u+1}}_{=\frac{1}{u+1}}\Big)^{q-1}\frac{1}{(u+1)^2}\,du =$$

$$= \int_0^\infty \frac{u^{p-1}}{(u+1)^{p-1+q-1+2}}\,du = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}$$

sedaj še DOKAZ IZREKA: $B(p,q) = \dfrac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$

$$\Gamma(p+q) = \int_0^\infty t^{p+q-1}e^{-t}\,dt = \int_0^\infty (1+u)^{p+q-1}s^{p+q-1}e^{-(1+u)s}(1+u)\,ds =$$

če t teče $0\to\infty$,
s teče $0\to\infty$

$t = \underbrace{(1+u)}_{\text{neka konstanta} > 0}\cdot s$

$dt = (1+u)\,ds$

$$= (1+u)^{p+q}\int_0^\infty s^{p+q-1}e^{-(1+u)s}\,ds =$$

sedaj to vrstico pomnožimo z $\dfrac{u^{p-1}}{(1+u)^{p+q}}$ in jo integriramo po u v mejah $0,\infty$:

$$\Gamma(p+q)\cdot\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\,du = \int_0^\infty u^{p-1}\left(\int_0^\infty s^{p+q-1} e^{-(1+u)s}\,ds\right)du$$

opazimo, da je na levi produkt $\Gamma(p+q)\cdot B(p,q)$, na desni pa moramo premisliti, da smemo obrniti vrstni red integracije – enakomerna konvergenca.

$$\Gamma(p+q)\cdot B(p,q) = \iint_0^\infty u^{p-1} s^{p+q-1} e^{-(1+u)s}\,du\,ds =$$

$$= \int_0^\infty s^{p+q-1} e^{-s}\left(\int_0^\infty u^{p-1} e^{-us}\,du\right)ds = \int_0^\infty s^{\cancel{p+}q-1} e^{-s}\left(\int_0^\infty \left(\frac{v}{\cancel{s}}\right)^{p-1} e^{-v}\frac{dv}{\cancel{s}}\right)ds$$

$v = us$, $s > 0$

$dv = s\cdot du$

$du = \frac{dv}{s}$

$$= \int_0^\infty s^{q-1} e^{-s}\left(\int_0^\infty v^{p-1} e^{-v}\,dv\right)ds =$$

$$= \int_0^\infty s^{q-1} e^{-s}\,\Gamma(p)\,ds = \Gamma(p)\Gamma(q) \quad \square$$

Funkciji Γ in B uporabljamo predvsem za integracijo preje raznanih določenih integralov

PRIMERa uporabe:

1. $$\int_0^{\frac{\pi}{2}} \sin^{2p-1}x\,\cos^{2q-1}x\,dx = I_{p,q}, \quad p,q>0$$

uvedemo $t = \sin^2 x \longrightarrow t^{\frac{1}{2}} = \sin x \longrightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-t} = (1-t)^{1/2}$

$dt = 2\sin x\cos x\,dx$

$$dx = \frac{1}{2}\,\frac{dt}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}$$

$$I_{p,q} = \int_0^1 \left(t^{\frac{1}{2}}\right)^{2p-1} \left(\left(1-t\right)^{\frac{1}{2}}\right)^{2q-1} \frac{dt}{2t^{1/2}(1-t)^{1/2}} = \frac{1}{2}\int_0^1 t^{p-1}(1-t)^{q-1}\,dt =$$

$$= \frac{1}{2} B(p,q)$$

KONKRETEN PRIMER:

let $p = n \in \mathbb{N}$, $q = m \in \mathbb{N}$

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1} x \cos^{2m-1} x \, dx = \frac{1}{2} B(n,m) = \frac{1}{2} \frac{\Gamma(n)\cdot\Gamma(m)}{\Gamma(n+m)} =$$

$$= \frac{1}{2} \frac{(n-1)!\,(m-1)!}{(n+m-1)!}$$

KONKRETEN PRIMER:

2. $$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma(1)}^{=1}} = \Gamma^2\left(\frac{1}{2}\right)$$

lema

$$\int_0^\infty \frac{u^{\frac{1}{2}-1}}{(1+u)^{1/2+1/2}} du = \int_0^\infty \frac{du}{\sqrt{u}\,(1+u)} = \int_0^\infty \frac{2t\,dt}{t(1+t^2)} = 2\int_0^\infty \frac{dt}{1+t^2} =$$

$$t = \sqrt{u} \rightarrow u = t^2$$

$$dt = \frac{1}{2\sqrt{u}} du$$

$$du = 2t\,dt$$

$$= 2 \operatorname{atan} t \Big|_0^\infty = \pi$$

SKLEP: $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

POMEMBNE VREDNOSTI: $$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt = \sqrt{\pi} = \int_0^\infty \left(\cancel{\frac{x^2}{2}}\right)^{-1/2} e^{-\frac{x^2}{2}} \cancel{x}\,dx =$$

$$/ \cdot \frac{1}{2\sqrt{\pi}}$$

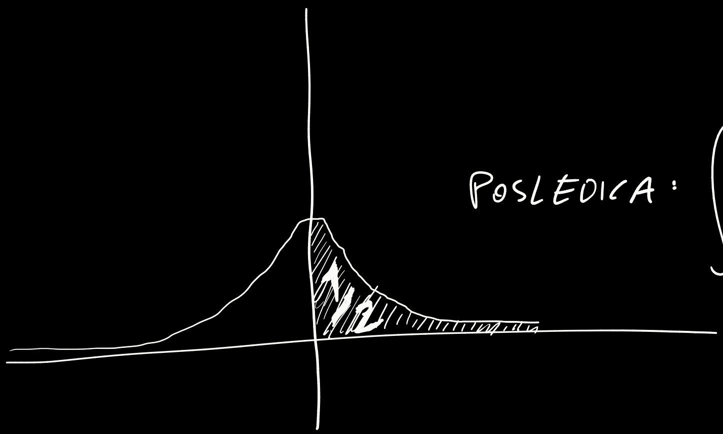
vpeljimo $t = \frac{x^2}{2}$ $dt = x\,dx$

ko teče t $0 \to \infty$, x teče tudi: $0 \to \infty$

$$= \sqrt{2}\int_0^\infty e^{-\frac{x^2}{2}} dx = \sqrt{\pi}$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{1}{2}$$

Pomen: $\rho(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, znana tudi kot Gaussova funkcija oziroma gostota std normalne porazdelitve slučajne spremenljivke.



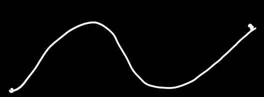
POSLEDICA: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = 1$

brez TRIKA tega integrala ni moč izračunati, saj je $\int e^{-x^2/2}\, dx$ znano neelementarna funkcija.

2. VECTERNI INTEGRAL

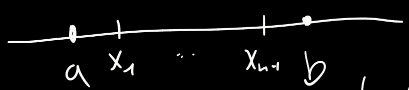
Cilj: Definirati integral omejene fje n spremenljivk na omejeni množici $D \subset \mathbb{R}^n$. Najbolj se bomo posvetili primeru, ko je $n=2$, t.j. $\iint_D f(x,y)\, dx\, dy$ za $D \subset \mathbb{R}^2$.

Za začetek pa se spomnimo, kako je tak integral definiran za $n=1$. Let $f:[a,b] \to \mathbb{R}$ omejena.



$D = \{a = x_0, \dots, x_n = b\}$ DELITEV

$x_0 < \dots < x_n$



vedno obstajata, ker f omejena: $M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$ $\quad m_j = \inf_{x \in [x_{j-1}, x_j]} f(x)$

$S(f,D) = \sum_{j=1}^{n} M_j (x_j - x_{j-1})$ $\quad s(f,D) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$

zgornja in spodnja Darbouxova vsota.

Def: f je integrabilna na $[a,b] \Leftrightarrow \sup_{\text{delitve } D} s(f,D) = \inf_{\text{delitve } D} S(f,D) =: \int_a^b f(x)\,dx$.