ANA2 VFMF 2024-05-29

LDESKK

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y^{(0)} = f(x)$$

1. poiščemo splošno rešitev PHLDESS~~K~~ z nastavkom

$$y = e^{\lambda x}$$

$$y' = \lambda e^{\lambda x} \quad \to \quad \text{karakteristični polinom}$$

$$\vdots$$

$$a_n \lambda^n + \dots + a_1 \lambda + a_0 = 0$$

$$y^{(n)} = \lambda^n e^{\lambda x}$$

case vse λ_i so paroma različne:

rešitev: $C_1 e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x} = y_H$

case λ je k-kratna ničla:

$$y_H = \underbrace{C_1 e^{\lambda x} + C_2 x e^{\lambda x} + \dots + C_k x^{k-1} e^{\lambda x}} + \underbrace{\dots}_{\text{ostale ničle}}$$

case $\mathrm{Im}(\lambda) \neq 0$:

$\Rightarrow$ tudi $\bar{\lambda}$ je ničla.

$$y_H = \dots + C_1 e^{\lambda x} + C_2 e^{\bar{\lambda} x} + \dots$$

spomnimo se, da $e^{a+ib} = e^a(\cos b + i \sin b)$

če je $\lambda = a+bi \Rightarrow e^{\lambda x} = e^{ax} \cos bx + i e^{ax} \sin bx$

par rešitev $e^{(a+bi)x}, e^{(a-bi)x}$ zamenjamo

z $e^{ax} \cos bx, e^{ax} \sin bx$

IV

Poišči splošno rešitev

$$y'''' + 6y'' + 9y = 0$$

polinom: $\lambda^4 + 6\lambda^2 + 9 = 0$. let $x = \lambda^2$

$$x^2 + 6x + 9 = 0$$

$$(x+3)^2$$

$$x = -3$$

$$\lambda_{1,2} = i\sqrt{3}$$

$$\lambda_{3,4} = -i\sqrt{3}$$

$$(\lambda - i\sqrt{3})(\lambda + i\sqrt{3})^2$$

$$y_H = C_1 e^{i\sqrt{3}x} + C_2 x e^{i\sqrt{3}x} + C_3 e^{-i\sqrt{3}x} + C_4 x e^{-i\sqrt{3}x}$$

toda to so kompleksne f-je

prvi par $\left(e^{i\sqrt{3}x},\ e^{-i\sqrt{3}x}\right) \longrightarrow \left(\cos(\sqrt{3}x), \sin(\sqrt{3}x)\right)$

drugi $\left(xe^{i\sqrt{3}x},\ xe^{-i\sqrt{3}x}\right) \longrightarrow \left(x\cos(\sqrt{3}x), x\sin(\sqrt{3}x)\right)$

Re Im

$$y_H = C_1 \cos(\sqrt{3}x) + C_2 x\cos(\sqrt{3}x) + C_3 \sin(\sqrt{3}x) + C_4 x\sin(\sqrt{3}x)$$

Kaj pa ko LDESKK ni homogena $(f(x) \neq 0)$

$$a_n y^{(n)} + \dots + a_0 y = f(x)$$

Rešimo PHLDESKK v splošno rešitev:

$$y_H = C_1 y_1(x) + \dots + C_n y_n(x)$$

$\hookrightarrow$ neke fje (exp/trig)

sedaj delamo variacijo konstant:

$$y_H = C_1(x)\, y_1(x) + \dots + C_n(x)\, y_n(x)$$

s sistemom:

$$\begin{bmatrix} y_1^{(1)} & \dots & y_n^{(1)} \\ \vdots & & \vdots \\ y_1^{(n)} & \dots & y_n^{(n)} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x)/a_n \end{bmatrix}$$

Reševanje LDESKK

$$a_n y^{(n)} + \dots + a_0 y = \underbrace{P(x)}_{\text{polinom}} e^{\lambda x}$$

y_p dobimo z nastavkom, če je f take oblike.

če je λ k-kratna ničla karakt. polinoma:

$$y_p = \underbrace{Q(x)}_{\text{polinom z neznanimi koef iste stopnje kot } P} x^k e^{\lambda x}$$

N: Poišči splošne rešitve $y'' - y = \boxed{2} e^{2x}$ — P stopnje 0, 0 ni ničla k. pol.

PHLDESKK: k. polinom: $x^2 - 1 = 0$

$(x-1)(x+1)$

$\lambda_1 = 1 \quad \lambda_2 = -1$

$C_1 e^{-x} + C_2 e^{x} =$

$$y_p = A x^0 e^{\lambda x} = A e^{2x}$$

pol. st. 0

$$y_p' = 2Ae^{2x}$$
$$y_p'' = 4Ae^{2x}$$

$$4Ae^{2x} - Ae^{2x} = 3Ae^{2x} = 2e^{2x}$$

$$\to A = 2/3 \Rightarrow y_p = \tfrac{2}{3} e^{2x}$$

$$y_{splošna} = y_H + y_p = C_1 e^{x} + C_2 e^{-x} + \tfrac{2}{3} e^{2x}$$ ✓

N

b.) $y'' - y = e^{x}$ $P = 1$, $\lambda = 1$

PRIDRUŽENA: $y'' - y = 0$ (kot prej)

$$y_H = C_1 e^{x} + C_2 e^{-x}$$

1 je ničla kratnosti 1 kar. pol.:

$$y_p = C x e^{x}$$

$$y_p' = Ce^{x} + Cxe^{x}$$

$$y_p'' = 2Ce^{x} + Cxe^{x}$$

$$2Ce^{x} + \cancel{Cxe^{x}} - \cancel{Cxe^{x}} = e^{x}$$

$$2C\cancel{e^{x}} = \cancel{e^{x}}$$

$$2C = 1$$

$$C = 1/2$$

$$y_p = \tfrac{1}{2} x e^{x}$$

$$y_{spl} = \tfrac{1}{2} x e^{x} + C_1 e^{x} + C_2 e^{-x}$$

N

$$y'' - y = e^{x} + 2e^{2x}$$

$$y_H = C_1 e^{x} + C_2 e^{-x} + \underbrace{\tfrac{1}{2} x e^{x}}_{y_p \text{ za } y''-y=e^{x}} + \underbrace{\tfrac{2}{3} e^{2x}}_{y_p \text{ za } y''-y=2e^{2x}}$$

N

Poišči splošno rešitev

cosinus hiperbolicus

$$y'' - 5y' + 6y = ch(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

za vsakega posebej y_p

kar. pol.: $x^2 - 5x + 6 = 0$

$$(x-2)(x-3) = 0$$

$$\lambda_1 = 2$$

$$\lambda_2 = 3$$

$$y_H = C_1 e^{2x} + C_2 e^{3x}$$

$$y_{p_1} = A x^0 e^{x} = Ae^{x}$$

$$y_{p_2} = B x^0 e^{-x} = Be^{-x}$$

$$y_{p_1}' = Ae^{x} \qquad y_{p_1}'' = Ae^{x}$$

$$y_{p_2}' = -Be^{-x} \qquad y_{p_2}'' = Be^{-x}$$

$$B\cancel{e^{-x}} + 5B\cancel{e^{-x}} + 6B\cancel{e^{-x}} = \frac{1}{2}\cancel{e^{-x}}$$

$$12B = \frac{1}{2}$$

$$B = \frac{1}{24}$$

$$Ae^{x} - 5Ae^{x} + 6Ae^{x} = \frac{1}{2}e^{x}$$

$$2A\cancel{e^{x}} = \frac{1}{2}\cancel{e^{x}}$$

$$2A = \frac{1}{2}$$

$$A = \frac{1}{4}$$

$$y_{spl} = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{4}e^{x} + \frac{1}{24}e^{-x}$$

N

$$y'' + 2y' + 5y = 17\sin(2x)$$

k.p.: $x^2 + 2x + 5 = 0$

$\lambda_1 = -1 + 2i$

$\lambda_2 = -1 - 2i$

$$y_H = C_1 e^{(-1+2i)x} + C_2 e^{(-1-2i)x} =$$

Re Im

$$= C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x)$$

$e^{i2x} = \cos 2x + i\sin 2x$

RAZŠIRIMO DE NA $\mathbb{C}$, $\operatorname{Im}(y_{spl})$ bo rešitev za $\sin 2x$

$$\tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = 17e^{i2x}$$

$$\Downarrow \quad \tilde{y}_p = A x^0 e^{i2x} = Ae^{i2x}$$

$$\tilde{y}_p' = i2Ae^{i2x} \qquad \tilde{y}_p'' = -4Ae^{i2x}$$

vstavi: $-4Ae^{i2x} + 4iAe^{i2x} + 5Ae^{i2x} = 17e^{i2x}$

$$A = \frac{17}{1+4i}$$

$$\tilde{y}_p = \frac{17}{1+4i} e^{2ix} \Longrightarrow$$

za ovig. enačbo

$$y_p = \operatorname{Im}(\tilde{y}_p) = \operatorname{Im}\left(\frac{17(1-4i)}{17} e^{2ix}\right) =$$

$$= \operatorname{Im}\left((1-4i)(\cos 2x + i\sin 2x)\right) =$$

$$= \operatorname{Im}\left(\cos 2x + i\sin 2x - 4i\cos 2x + 4\sin 2x\right) =$$

$$= \sin 2x - 4\cos 2x$$

$$y_{spl} = \underbrace{C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x)}_{y_H} \underbrace{-4\cos(2x) + \sin(2x)}_{y_P}$$

[EULER-CAUCHYJEVA dif. en.]

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_0 y = f(x)$$

1. rešimo homogeno $(f(x)=0)$

Rešitve bodo x^λ. vstavi $y = x^\lambda$:

kar. pol.: $a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$

ničle ... λ_i

case λ_i paroma različne:

$$y_H = C_1 x^{\lambda_1} + \dots + C_n x^{\lambda_n}$$

case λ k-kratna ničla:

$$x^\lambda,\ x^\lambda \ln x,\ \dots,\ x^\lambda (\ln x)^{k-1}$$

case $\lambda = a+ib \Rightarrow \bar{\lambda} = a-ib$ je spet ničla

$$x^{a+bi},\ x^{a-bi}$$

$$\Downarrow$$

$$x^a \cos(b \ln x),\ x^a \sin(b \ln x)$$

2. tomat: y_p: $f(x) = P(\ln x)\, x^{\lambda}$

polinom v spremenljivki $\ln x$

če je λ k-kratna ničla:

$$y_p = Q \ln x\, (\ln x)^k\, x^{\lambda}$$

iste stopnje kot P

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1. Poišči splošno rešitev:

$$x^2 y'' - x y' + 2y = 1 + (\ln x)^2 = P(\ln x) \cdot x^0$$

druge stopnje: $1 + x^2$

i. $x^2 y'' - x y' + 2y = 0$

$y = x^{\lambda}$

$y' = \lambda x^{\lambda - 1}$

$y'' = \lambda(\lambda - 1) x^{\lambda - 2}$

$x^2 \lambda(\lambda-1) x^{\lambda-2} - x \lambda x^{\lambda-1} + 2x^{\lambda} = 0$

$\lambda(\lambda-1) x^{\lambda} - \lambda x^{\lambda} + 2x^{\lambda} = 0 \quad / : x^{\lambda}$

$\lambda(\lambda-1) - \lambda + 2 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda_{1,2} = 1 \pm i$

$y_H = C_1 x^{1+i} + C_2 x^{1-i}$

$y_H = C_1 x^1 \cos \ln x + C_2 x^1 \sin \ln x$

$$y_p = \left(A + B\ln x + C(\ln x)^2\right)(\ln t)^0\, x^0 = A + B\ln x + C(\ln x)^2$$

ker je 0-kratna ničla

$y_p' = \frac{B}{x} + 2C(\ln x)\frac{1}{x}$

$y_p'' = -B/x^2 + 2C\,1/x^2 - 2C(\ln x)\frac{1}{x^2}$

vstavi v enačbo:

$$\cancel{x^2}\left(-B/\cancel{x^2} + 2C\frac{1}{\cancel{x^2}} - 2C(\ln x)\frac{1}{\cancel{x^2}}\right) - \cancel{x}\left(\frac{B}{\cancel{x}} + 2C(\ln x)\frac{1}{\cancel{x}}\right) +$$

$$+2\left(A + B\ln x + C(\ln x)^2\right) = 1 + (\ln x)^2$$

$$-B + 2C - 2C(\ln x) - B - 2C\ln x + 2A + 2B\ln x + 2C(\ln x)^2 =$$
$$= 1 + (\ln x)^2$$

$$-B + 2C - B + 2A = 1$$
$$-2C - 2C + 2B = 0 = -2 + 2B$$
$$2C = 1 \qquad C = \frac{1}{2} \qquad 2B - 2 = 0$$
$$2B = 2$$
$$B = 1$$

$$-1 + 1 - 1 + 2A = 1$$
$$2A = 2$$
$$A = 1$$

$$y_{splo} = 1 + \ln x + \frac{(\ln x)^2}{2} + C_1 x\cos\ln x + C_2 x\sin\ln x$$

[SISTEMI dif. en.]

Iščemo $x(t)$, $y(t)$ t.:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$
$$\Downarrow$$

zdb iščemo vektorsko f-jo $x(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Longleftrightarrow \dot{\vec{x}} = A\vec{x}$$

(i) določimo lastv A

določimo lastne A

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t} \begin{bmatrix} v_1 \end{bmatrix} + C_2 e^{\lambda_2 t} \begin{bmatrix} v_2 \end{bmatrix}$$

Komplikacija: če ima A dvojno lastv λ in zanjo le en LN lastne.
(A se ne da diagonalizirati)

$\lambda \longrightarrow v$... lastni vektor, nato določimo še korenski vektor k tj: $(A-\lambda I)k = v$

in tedaj

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda t} \begin{bmatrix} v \end{bmatrix} + C_2 e^{\lambda t} \left(\begin{bmatrix} k \end{bmatrix} + t \begin{bmatrix} v \end{bmatrix} \right)$$

N

Poišči splošno rešitev sistema

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Delta_A(x) = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

$\lambda_1 = 1$:

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdots v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdots v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

N

Poišči spl. rešitev $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Delta_A = (3/2 - \lambda)(\tfrac{1}{2} - \lambda) + \frac{1}{4} = \frac{3}{4} - \frac{3}{2}\lambda - \frac{1}{2}\lambda + \lambda^2 + \frac{1}{4} =$$

$$= 1 - \lambda + \lambda^2 = (\lambda - 1)(\lambda - 1)$$ dvojna lastv!

$$\ker \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ le en lastv!

potrebujemo korenski vektor k:

$$\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$k = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{t} \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$