Eksaktne diferencialne enačbe.

motivacija: Družina krivulj, dana implicitno z

$$u(x,y) = C$$

$\Downarrow$ katero DE zadoščajo?

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y' = 0 \qquad / y' = \frac{dy}{dx}$$

$$\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$$

$$\frac{du}{dx} + \frac{du}{dx} = 0$$

Če imamo DE $\quad y' = f(x,y) = -\frac{P(x,y)}{Q(x,y)}$:

$$P(x,y) + Q(x,y)\,y' = 0 \qquad / y' = \frac{dy}{dx}$$

$$\boxed{P(x,y) + Q(x,y)\frac{dy}{dx} = 0} \qquad \text{če je } P = \frac{du}{dx} \text{ in } Q = \frac{du}{dy}$$

$\Downarrow$ rešitve DE so nivojnice u

$$u(x,y) = C$$

kako najti u?

odvajamo P po y in Q po x, dobimo u_{xy} in u_{yx}, ki morata biti (za zvezne f-je).

Primer: Poišči splošno rešitev

$$\underbrace{(2x-y)}_{P(x,y)}\,dx - \underbrace{(x+y)}_{Q(x,y)}\,dy = 0$$

opomba: $y' = \frac{2x-y}{x+y}$

a velja $\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x}$?

$P_y = -1 \qquad Q_x = -1 \qquad \checkmark$

poiščimo u, da je

$$u_x = 2x^3 - y \Rightarrow u = \frac{1}{2}x^4 - xy + \varphi(y)$$

$$u_y = -x - y$$

odvajaj po y in enači z u_y

$$u_y = -x + \varphi'(y) = -x - y$$

$$\varphi'(y) = -y$$

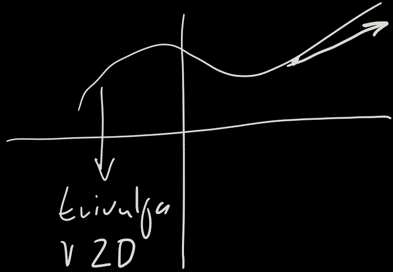
$$\varphi(y) = -\frac{y^2}{2} + C$$

torej $u = \frac{1}{2}x^4 - xy - \frac{y^2}{2} + C$ je splošna rešitev.

EDE :: geometrijski pomen:

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$P(x,y) + Q(x,y)\,y' = 0$$

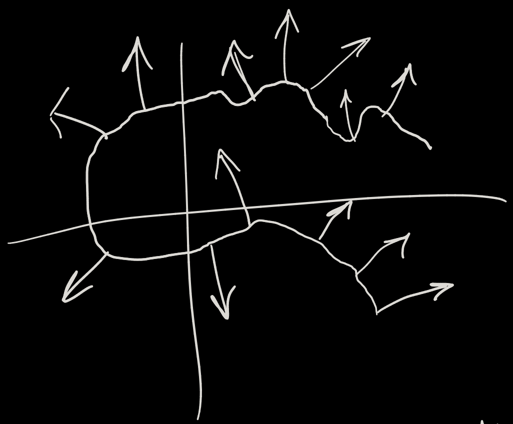


$(1, y')$ v točki $(x, y(x))$

Pogoj za rešitev je, da je v vsaki točki pravokotna na vektorsko polje (P,Q):

$$(P,Q)\cdot(1,y') = 0$$

$$(P,Q) \perp (1,y')$$



tak pa če

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \ ?$$

Poiščemo integrirajoči množitelj $\mu(x,y)$, da je

$$\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x}$$

torej za u velja

$$u_x = \mu P$$
$$u_y = \mu Q$$

predstavlja zgolj skaliranje vektorjev v vektorskem polju

najti tak μ je težko!

N

$$\underbrace{(2xe^x + e^x - y^2)}_{P}\,dx \underbrace{-2y}_{Q}\,dy = 0$$

NASVET: izbereš lahko μ, odvisen le od X (*)

$$\frac{\partial P}{\partial y} = -2y$$

$$\frac{\partial Q}{\partial x} = 0^{(**)}$$

Q

$$\frac{\partial \mu \times P}{\partial y} = \frac{\partial \mu \times Q}{\partial x}$$

$\to 0$ (*) $\qquad\qquad$ 0 (**)

$$\frac{\partial \mu}{\partial y} P + \frac{\partial P}{\partial y}\mu = \frac{\partial \mu}{\partial x} Q + \frac{\partial Q}{\partial x}\mu$$

$$\cancel{-2y}\,\mu = \mu'\cancel{(-2y)}$$

$$\mu = Ae^x, \text{ vzemimo } A = 1, \qquad \mu(x) = e^x$$

$(***)$

$u_x = e^x P = e^x(2xe^x + e^x - 2y^2)$

$u_y = e^x Q = e^x(-2y)$

$\hookrightarrow u = \int e^x(-2y)\,dx = -2y\int e^x dx = -e^x 2y + \varphi(x)$

odvajaj po x in enačiš u_x od prej $(**)$

$e^x(2xe^x + e^x - 2y^2) = -e^x 2y + \varphi'(x)$

$2xe^{2x} + e^{2x} - \cancel{2y^2 e^x} = \cancel{-e^x 2y} + \varphi'(x)$

$\varphi'(x) = e^x(2xe^x + e^{2x})$

N

Poišči splošno rešitev:

$$y(x^2+y^2+1)dx - x(x^2+y^2+1)dy = 0$$

nasvet: $\mu(x,y) = f(xy)$

$$P = y(x^2+y^2+1)$$

$$Q = -x(x^2+y^2-1)$$

$$P_y = (x^2+y^2+1) + y(2y) =$$
$$= x^2+3y^2+1$$

$$Q_x = -(x^2+y^2-1) - x(2x) =$$
$$= -3x^2-y^2+1$$

iščemo f:

$$\frac{\partial(f(xy)\,P)}{\partial y} = \frac{\partial(f(xy)\,Q)}{\partial x}$$

$$f'(xy)\,x\cdot P + f(xy)\,P_y = f'(xy)\,y\,Q + f(xy)\,Q_x$$

$$f'(xy)\,(x\cdot P - y\cdot Q) = f(xy)\,(Q_x - P_y)$$

$$f'(xy)\left(x\left(y\left(x^2+y^2+1\right)\right) - y\left(-x\left(x^2+y^2-1\right)\right)\right) =$$
$$f(xy)\left((-3x^2-y^2+1) - (x^2+3y^2+1)\right) =$$
$$= f(xy)\left(-4x^2-4y^2\right)$$

$$f'(xy)\left(2x^3y+2xy^3\right) = \left(-4x^2-4y^2\right)f(xy)$$

$$f'(xy)\,2xy\,\cancel{(x^2+y^2)} = -4\,\cancel{(x^2+y^2)}\,f(xy)$$

$$2xy\cdot f'(xy) = -4\,f(xy)$$

naj bo $t = xy$:

$$t f' + 2f = 0$$ spet DE!

$$t f' = -2f$$

$$t \frac{\partial f}{\partial t} = -2f$$

$$\int \frac{t}{\partial t} = \int \frac{-2f}{\partial f}$$

$$\ln|t| = -2 \ln|f| \cancel{+C}$$

$$|f| = e^{-2\ln|t|}$$

$$e^{\ln|t|^{-2}} = |f|$$

$$f = t^{-2}$$

$$\mu(xy) = f(xy) = \frac{1}{x^2y^2}$$

$$u_x = \mu(xy) \cdot P = \frac{1}{x^2y^{\cancel{2}}}\left(\cancel{y}(x^2+y^2+1)\right) = \frac{1}{y} + \frac{y}{x^2} + \frac{1}{x^2y}$$

$$u_y = \mu(xy) \cdot Q = \frac{1}{x^{\cancel{2}}y^2}\left(-\cancel{x}(x^2+y^2-1)\right) = \frac{-x}{y^2} - \frac{1}{x} + \frac{1}{xy^2}$$

integr. po x

$$u = \frac{x}{y} - yx^{-1} - \frac{x^{-1}}{y} + \varphi(y)$$

odvajamo po y in enačimo z u_y:

$$\cancel{\frac{-x}{y^2}} - \cancel{x^{-1}} + \cancel{\frac{x^{-1}}{y^2}} + \varphi'(y) = \cancel{\frac{-x}{y^2}} - \cancel{\frac{1}{x}} + \cancel{\frac{1}{xy^2}}$$

$$\varphi'(y) = 0 \Rightarrow \varphi(y) = C$$

$$\Rightarrow u(x,y) = \frac{x}{y} - \frac{y}{x} - \frac{1}{xy} + C$$

rešitve so

$$\frac{x}{y} - \frac{y}{x} - \frac{1}{xy} = D$$

DEVR:

$y'' = f(x, y, y')$ → splošna rešitev ima dva konstantna parametra

$y = f(x, C_1, C_2)$

da dobimo eno rešitev, pa naloga poda 2 začetna pogoja

$$y(x_0) = y_0 \quad \text{in} \quad y'(x_0) = p_0$$

N

Reši splošno rešitev:

$$\boxed{x^2 y'' = y'^2} \qquad y'' = \frac{y'^2}{x^2}$$

→ tule y ne nastopa, zato uvedemo $z = y'$:

NIŽANJE REDA!

$$x^2 z' = z^2$$

$$x^2 \frac{\partial z}{\partial x} = \frac{z^2}{dz}$$

$$\int \frac{x^2}{\partial x} = \int \frac{z^2}{dz}$$

$$\int \frac{1}{x^2} dx = \int \frac{1}{z^2} dz \quad \ldots$$

N

$$y'' = f(y, y')$$ (x ne nastopa eksplicitno)

uvedemo $v(y) = y' \quad / \frac{d}{dx}$

$$v'(y)\, y' = y''$$

$$\Rightarrow y'' = v v'$$

$$\Rightarrow v v' = f(y, v)$$ enačba 1. reda za $v(y)$

N

Reši $y y'' = 2 y y' - y'^2$

let $v = y'$, $v'(y)\, y' = y''$

$$v' \cdot v = y''$$

$$y(v' v) = 2 y v - v^2$$

$$y v' \cancel{v} = 2 y \cancel{v} - v^{\cancel{2}}$$

$$y v' = 2y - v$$

LDE

$$y v' + v = 2y$$

PHLDE $y v' + v = 0$

$$-y v' = v$$

$$-y \frac{\partial v}{\partial y} = v$$

$$\int \frac{-y}{dy} = \int \frac{v}{\partial v}$$

$$-\ln|y| = \ln|v| + \ln c$$

$$v_H = C y^{-1} = \frac{C}{y}$$

$$v_p = C(y)\, y^{-1}$$

$$v_p' = C'(y)\, y^{-1} - C(y)\, y^{-2}$$

vstavi v LDE:

$$y\left(c'(y)\,y^{-1} \cancel{- c(y)\,y^{-2}}\right) \cancel{+ c(y)\,y^{-1}} = 2y$$

$$c'(y) = 2y$$

$$c(y) = y^2$$

$$V_{spl.} = Cy^{-1} + y$$

...

N

LDE 2. reda

$$y'' + p(x)y' + q(x)y = r(x)$$

1. korak: splošna rešitev PHLDE2R

$$y'' + p(x)y' + q(x)y = 0 \quad (*)$$

ni splošnega postopka

nato velja $y_H = C_1 y_1 + C_2 y_2$

2. korak: partikularna rešitev LDE2R

$$y_p = C_1(x)\,y_1 + C_2(x)\,y_2$$

$$\Downarrow$$

$$C_1, C_2 = ?$$

rešimo sistem enačb

$$C_1' y_1 + C_2' y_2 = 0$$

$$C_1' y_1' + C_2' y_2' = r(x)$$

da dobimo C_1, C_2

3. korak $y_{spl} = y_H + y_p$

(*) kako dobiti y_1, y_2 za primer 2?

recimo, da imamo $y'' + py' + qy = r$ in eno rešitev.

$\Downarrow$

s pomočjo determinante $\boxed{\text{Wronskega}}$ lahko določimo y_2

$$W(x) = e^{-\int p(x)\,dx}$$

$\Downarrow$

če sta y_1, y_2 dve LN rešitvi $y'' + py' + qy = 0$, potem

velja $\underbrace{W(x) = y_1 y_2' - y_2 y_1'}$

LDE1R za y_2.

N

Poišči splošno rešitev $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$, če veš, da je $y_1 = x^{-1/2} \cdot \sin x$ ena od rešitev.

$$y'' + \underbrace{x^{-1}}_{p(x)} y' + x^{-2}(x^2 - \tfrac{1}{4}) y = 0$$

$$W(x) = e^{-\int x^{-1}\,dx} = e^{-\ln|x|} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|}$$

$$\underline{W(x) = \frac{1}{x}}$$

$$y_1 y_2' - y_2 y_1' = W(x)$$

$$y_1' = -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x$$

$$\underbrace{x^{-1/2}\sin x\, y_2' - y_2\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right) = \frac{1}{x}}_{\text{LDE1R za } y_2}$$

PHLDE1R:

$$x^{-1/2}\sin x\, y_2' - y_2\left(-\frac{1}{2}x^{3/2}\sin x + x^{-1/2}\cos x\right) = 0$$

$$x^{-1/2}\sin x\, y_2' = y_2\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right)$$

$$\parallel$$

$$x^{-1/2}\sin x\,\frac{dy_2}{dx}$$

$$\frac{x^{-1/2}\sin x}{-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x}\cdot\frac{1}{dx} = \frac{y_2}{dy_2}$$

$$\int \frac{-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x}{x^{-1/2}\sin x}\,dx = \int \frac{1}{y_2}\,dy$$

$$C + \ln|y_2| = \int \frac{-x^{-1}x^{-3/2}\cancel{\sin x}}{x^{-1/2}\cancel{\sin x}}\,dx + \int \frac{\cancel{x^{-1/2}}\cos x}{\cancel{x^{-1/2}}\sin x}\,dx =$$

$$= -\int \frac{1}{2}x^{-1}\,dx + \ln|\sin x| = -\frac{1}{2}\ln|x| + \ln|\sin x|$$

$$\Rightarrow \ln|y_2| = -\frac{1}{2}\ln|x| + \ln|\sin x|$$

$$\ln|y_2| = \ln|x|^{-1/2} + \ln|\sin x| + C$$

$$\underbrace{y_2 = x^{-1/2} \cdot \sin x \cdot D}_{\text{spec. reš. homog.}}$$

partik.:

$$y_p = x^{-1/2} \sin x \, D(x)$$

$$y_p' = -\frac{1}{2} x^{-3/2} \sin x \, D(x) + x^{-1/2} \cos x + x^{-1/2} \sin x \, D'(x)$$

vstavimo:

...

$$D'(x) = \frac{1}{\sin^2 x}$$

$$D(x) = \frac{1}{\cot x}$$

$$y_z = y_s + y_p$$

... obupam :(

LDESKK

$$a_n y^{(n)} + \dots + a_0 y = f(x)$$

1. korak: poišči splošno rešitev PHLDESKK, t.j. $f(x) = 0$

nastavek: rešitve so $y = e^{x\lambda}$

$$y' = \lambda e^{\lambda x}$$

$$\dots$$

$$y^{(n)} = \lambda^n e^{\lambda x}$$

$$\Downarrow$$

$$\underbrace{a_n \lambda^n e^{\lambda x} + \dots + a_0 = 0}_{\text{karakteristični polinom}}$$

če so vse λ_i paroma različne, je pologutna rešitev

$$y_p = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \dots + C_n e^{\lambda_n x}$$

PRIMER:

N Reši splošne rešitve enačb

a.) $y'' - y = 0$

polinom:

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

splošna rešitev je

$$y_H = A e^{x} + B e^{-x}$$

b.) $y''' + y'' - 2y' = 0$

$$\lambda^3 + \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda^2 + \lambda - 2) = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = -2$$

$$\lambda_3 = 1$$

$$y_{spl} = A + Be^{-2x} + Ce^{x}$$

KAJ PA, ČE JE λ k-kratna rešitev polinoma?

tedaj je splošna rešitev

$$e^{\lambda x},\ e^{\lambda x}x,\ e^{\lambda x}x^{2},\ \dots,\ e^{\lambda x}x^{k-1}$$

Primer:

i.) $y''-2y'+y=0$

$\lambda^2-2\lambda+1=0$

$\lambda_1=1$

$\lambda_2=1$

$y_{spl}=C_1e^x+C_2xe^x$