Diferencialne enačbe:

① Poišči rešitev $xy' - y = x\,tg\left(\frac{y}{x}\right)$, ki zadošča $y(3) = 8\pi$

$$xy' = x\,tg\frac{y}{x} + y$$

$$\boxed{y' = \frac{y}{x} + tg\frac{y}{x}}$$ homogena dif. en.

uvedemo novo $z = \frac{y}{x}$

$$y = x \cdot z$$

$$y' = z + xz'$$

$$\cancel{z} + xz' = \cancel{z} + tg\,z$$

$$xz' = tg\,z$$

$$\frac{z'}{tg\,z} = \frac{1}{x}$$

$$\int \frac{\cos z}{\sin z}\,dz = \int \frac{1}{tg\,z}\,dz = \int \frac{1}{x}\,dx$$

$$\ln|\sin z| = \ln|x| + C$$

$$|\sin z| = e^{\ln|x| \cdot C}$$

$$\sin z = \pm e^{C} \cdot e^{\ln|x|}$$

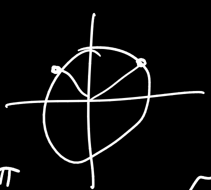
$$\sin z = \pm e^{C} \cdot |x| \qquad D \in \mathbb{R}$$

pozit. $\qquad \pm x$

$\sin x = A$

$x_1 = asin\,A + k2\pi$

$x_2 = \pi - asin\,A + 2k\pi$



$$\sin z = D \cdot x$$

$$\sin\frac{y}{x} = D\,x$$ splošna rešitev v implicitni obl.

$$\frac{y}{x} = asin(D\,x) + 2k\pi \qquad k \in \mathbb{Z}$$

ali $$\frac{y}{x} = \pi - asin(Dx) + 2k\pi$$

$$y = x(asin(Dx) + 2k\pi)$$

liha

$$y = x(\pi - asin(Dx) + 2k\pi)$$

$\to y = x\left(\pi + \operatorname{asin}(-Dx) + 2k\pi\right)$

$\to y = x\left(\operatorname{asin}(Dx) + k\pi\right) \quad k \in \mathbb{Z}$

sedaj še upoštevamo pogoj $y(3) = \pi 8$

Greno s postuSanjem:

$k = 0 : \quad x \operatorname{asin}(Dx)$

$8\pi = 3 \underbrace{\underbrace{\operatorname{asin}(D3)}_{\in [-\frac{\pi}{2}, \frac{\pi}{2}]}}_{\in [-\frac{3\pi}{2}, \frac{3\pi}{2}]}$ ne gre.

$k = 1 : \quad y = x(\operatorname{asin} Dx + \pi)$

$8\pi = \underbrace{3 \operatorname{asin} 3D + 3\pi}_{\in [-\frac{3\pi}{2} + 3\pi, \frac{3\pi}{2} + 3\pi]}$ ne gre

$k = 2 : \quad y = x(\operatorname{asin} Dx + 2\pi)$ $\quad 8\pi \notin$

$8\pi = 3 \operatorname{asin} 3D + 6\pi \quad \in [\frac{9\pi}{2}, \frac{15\pi}{2}]$

$k = 3 : \quad y = x(\operatorname{asin} Dx + 3\pi)$

$8\pi = 3 \operatorname{asin}(3D) + 9\pi$ gre ✓

$-\pi = 3 \operatorname{asin}(3D)$

$\operatorname{asin} 3D = -\frac{\pi}{3}$

$3D = \sin -\frac{\pi}{3} = \frac{-\sqrt{3}}{2}$

$D = -\frac{\sqrt{3}}{D}$

Končna rešitev:

$$y = x\left(\operatorname{asin}\left(-\frac{\sqrt{3}}{6}x\right) + 3\pi\right)$$

LINEARNE DIFERENCIALNE ENAČBE

$$p(x)y' + q(x)y = r(x)$$

1. korak: najdi splošno rešitev pripadajoče homogene linearne diferencialne enačbe, t.j. take, da je $r(x) = 0$

$$p(x)y' + q(x)y = 0$$

$$\frac{p(x)\,dy}{dx} = q(x)\,y$$

$$p(x)\,dy\;y = q(x)\,dx$$

$$\int_y dy = \int \frac{q(x)}{p(x)}\,dx$$

$$\ln|y| = F(x) + C$$

$$|y| = e^{Fx} + e^{c}$$

$$y = \pm e^{C}\,e^{F(x)}$$

$$y_H = D \cdot \varphi(x)$$

2. korak: poišči partikularno rešitev orig. enačbe

neznana

z nastavkom: $y_p = D(x) \cdot \varphi(x)$ vstavi v

$$py' + qy = r$$

dobimo pogoj za $D(x)$

dobimo eno $D(x)$ in

$y_p = D(x) \cdot \varphi(x)$.

$y_{splošna} = \underset{\in \mathbb{R}}{D}\varphi(x) + D(x)\,\varphi(x)$

N

Poišči splošno rešitev:

$$xy' - 3y = x$$

$$xy' - 3y = 0$$

$$xy' = 3y$$

$$x\frac{dy}{dx} = 3y$$

$$\frac{x}{dx} = 3y/dy$$

$$\int \frac{3}{x}\,dx = \int \frac{1}{y}\,dy$$

$$\ln|y| = 3\ln|x| + c$$

$$|y| = e^{\ln|x|^3} \cdot e^c$$

$$y = \pm e^{\ln|x|^3} \cdot e^c$$

$$y = \pm |x|^3 \cdot e^c$$

$$y = \pm(\pm x^3 \cdot e^c) = Dx^3 = y_H$$

2. korak: y_H

$$y_H = D(x)\cdot x^3 \qquad y_H' = D'(x)\cdot x^3 + D(x)\cdot 3x^2$$

vstavimo v $xy' - 3y = x$

$$x\left(D'(x)\cdot x^3 + D(x)\cdot 3x^2\right) - 3D(x)\cdot x^3 = x$$

$$D'(x)\cdot x^4 + \cancel{D(x)\,3x^3} - \cancel{3D(x)\cdot x^3} = x$$

$$D'(x)\cdot x^4 = x$$

$$D'(x) = x^{-3}$$

$$D(x) = \int x^{-3}\,dx = \frac{x^{-2}}{-2} =$$

$$= -\frac{1}{2x^2} = D(x)$$

vstavi

$$y_p = D(x)\cdot x^3 = \frac{-1}{2x^2}\cdot x^3 =$$

$$= \frac{-x}{2}$$

$$y_s = D x^3 - \frac{x}{2}$$

$$y_s = y_H - y_p$$

najdi katomo rešitev, za katero je $y(0) = 0$;

$D \in \mathbb{R}$ ⟶ vse

N

Poišči splošno rešitev: $(e^x+1)y' + e^x y = e^x - 1$

1. rešitev PHLDE:

$$(e^x+1)y' + e^x y = 0$$

$$\frac{(e^x+1)\,dy}{dx} = -e^x y$$

$$\frac{(e^x+1)}{-e^x dx} = \frac{y}{dy}$$

$$\int \frac{-e^x dx}{e^x+1} = \int \frac{dy}{y}$$

$$\ln|y| = -\ln(e^x+1) + C$$

$$|y| = e^{\ln(e^x+1)^{-1}} e^C$$

$$y_H = \cancel{\pm} D \cdot (e^x+1)^{-1}$$

2. korak: partikularna rešitev:

$$y_H = D(x)(e^x+1)^{-1}$$

$$y_H' = D'(x)(e^x+1)^{-1} - D(x)\frac{e^x}{(e^x+1)^2}$$

vstavimo v izvorno enačbo iz navodila:

$$(e^x+1)\left(D'(x)(e^x+1)^{-1} - D(x)(e^x+1)^{-2}e^x\right) + e^x\left(D(x)(e^x+1)^{-1}\right) = e^x-1$$

$$\cancel{(e^x+1)} D'(x)\cancel{(e^x+1)^{-1}} - \cancel{(e^x+1) D(x)(e^x+1)^{-2} e^x + e^x D(x)(e^x+1)^{-1}} = e^x - 1$$

$$D'(x) = e^x - 1$$

$$D(x) = \int (e^x - 1)dx = e^x - x$$

$$y_P = (e^x - x)(e^x+1)^{-1} = \frac{(e^x - x)}{e^x+1}$$

$$y_{SP} = \frac{D}{e^x+1} + \frac{e^x - x}{e^x+1} = \frac{D + e^x - x}{e^x+1}$$

BERNOULLIJEVA DIF. EN.

$p(x)y' + q(x)\,y = r(x)\,y^{\alpha}$ $(\alpha \in \mathbb{R})$, α, p, q podani

$\downarrow : y^{\alpha}$

$p y^{-\alpha} y' + q(x)\, y^{1-\alpha} = r$

uvedemo novo funkcijo

$z = y^{1-\alpha}$,

$z' = (1-\alpha) y^{-\alpha} \cdot y'$

$\downarrow$ zamenjamo z z in z'

$$\frac{p}{1-\alpha} z' + qz = r$$

LDE

N

reši $3y' + 2y = (1 + 3e^{x})\,y^{4}$ ... $\alpha = 4$ tukaj.

$\downarrow : y^{4}$

$z = y^{-3}$

$z' = -3\, y^{-4} y'$

$$3y' y^{-4} + 2y^{-3} = 1 + 3e^{x}$$

$$-z' + 2z = 1 + 3e^{x}$$

$$-z' + 2z = 0$$

$$2z = \frac{dz}{dx}$$

$$\int 2\,dx = \int \frac{1}{z}\,dz$$

$$\ln|z| = 2x + C$$

$$|z| = e^{2x+C} = e^{2x}\, e^{C}$$

$z = e^{2x} \cdot D$

lastiti:

$$z_p = D(x) \cdot e^{2x}$$

$$z'_p = D'(x) e^{2x} + D(x) e^{2x} \cdot 2$$

$$-D'(x)e^{2x} \cancel{- D(x)e^{2x} \cdot 2 + 2D(x)e^{2x}} = 1 + 3e^{x}$$

$$-D'e^{2x} = 1 + 3e^{x}$$

$$D'(x) = \frac{1+3e^{x}}{-e^{2x}}$$

$$D(x) = -\int \frac{1+3e^x}{e^{2x}} dx = -\int \frac{1}{e^{2x}} dx - \int \frac{3e^{x}}{e^{2x}} dx$$

$$= -\int e^{-2x} dx - 3\int e^{-x} dx =$$

$$= \frac{e^{-2x}}{-2} + 3e^{-x} = \frac{1}{2e^{2x}} + \frac{3}{e^{x}} = D(x)$$

$$z_p = \left(\frac{1}{2e^{2x}} + \frac{3}{e^{x}}\right) e^{2x} = \frac{1}{2} + 3e^{x}$$

$$z_{splošna} = De^{2x} + \frac{1}{2} + 3e^{x}$$

želimo pa y, torej: $z = y^{-3} \Rightarrow y = z^{-1/3}$

$$y_{spl} = \frac{1}{\sqrt[3]{De^{2x} + \frac{1}{2} + 3e^{x}}}$$

Doslej smo imeli dif. en. oblike $y' = f(x,y)$. z rešitvami

eksplicitno $y = \varphi(x, C)$ oz implicitno

družina krivulj $F(x, y, C) = 0$

OBRATEN PROBLEM:

kateri diferencialni enačbi zadošča družina krivulj

$$F(x,y,C)=0\ ?$$

$$\boxed{F(x,y,C)=0} \quad /\frac{d}{dx}$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot y'=0$$

izrazimo $C=\varphi(x,y)$, vstavimo in dobimo

$$y'=f(x,y)$$

Primer take začetka: katero dif. en. zadošča družina

$$y+\frac{x}{2}-Cx^3=0 \quad /\frac{d}{dx}$$

$$y'+\frac{1}{2}-3Cx^2=0$$

vstavi

izrazi C: $\frac{y+\frac{x}{2}}{x^3}=C$

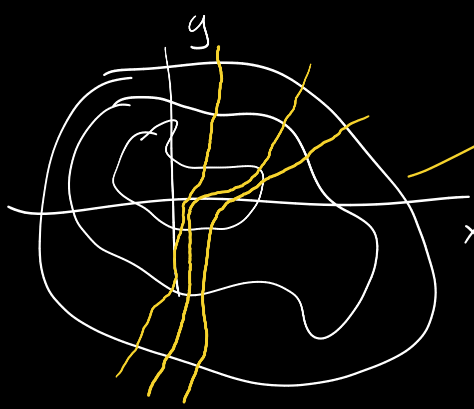
$$y'+\frac{1}{2}-3\,\frac{y+\frac{x}{2}}{x^3}\,x^2=0 \quad /\cdot x$$

$$xy'+\frac{x}{2}-3y-\frac{3x}{2}=0$$

$$xy'-3y=x \qquad \text{REŠENO.}$$

ISKANJE ORTOGONALNIH TRAJEKTORIJ :

Imejmo dano družino krivulj $F(x,y,C)=0$



različne izohipse za različne C-je.

Želimo poiskati družino krivulj, ki vsako krivuljo $f(x,y,C)=0$ seka pod kotom 90°.

Kako poiskati te tako imenovane ORTOGONALNE krivulje?

$F(x,y,C)=0 \Longrightarrow$ poiščemo dif. en., ki ji ta družina zadošča $\Rightarrow$ $y' = f(x,y)$. Tedaj

naklon je koeficient $f(x,y)$

Če se spomnimo, za premice velja

$y = -\frac{1}{k}x$ je pravokotna premici $y = kx$

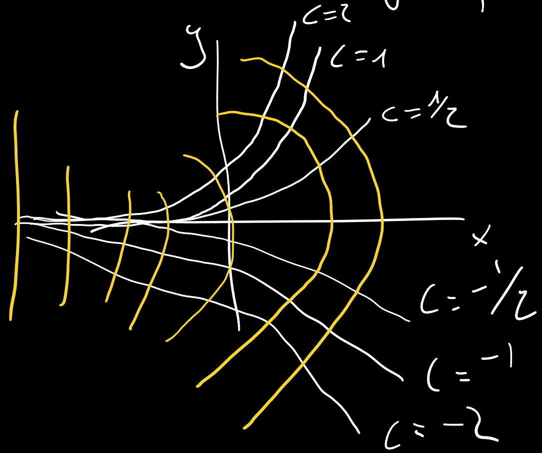
ortogonalna krivulja

↳ kakšni dif. enačbi zadošča?

veljati mora: $y' = \dfrac{-1}{f(x,y)}$

N

Dana je družina funkcij $y = Ce^x$. Določi družino ortogonalnih krivulj glede na



1. korak: najdi dif. en. iz družine

$$y = Ce^x$$

$$y' = Ce^x \qquad C = \frac{y}{e^x}$$

$$y' = \frac{y}{\cancel{e^x}} \cancel{e^x}$$

$$\underline{\underline{y' = y}}$$

Drugi korak: Dif. en. za ortog. krivulje:

$$\underline{\underline{y' = \frac{1}{f(x,y)} = \frac{-1}{y}}}$$

rešimo dif. en. $y' = \frac{-1}{y}$

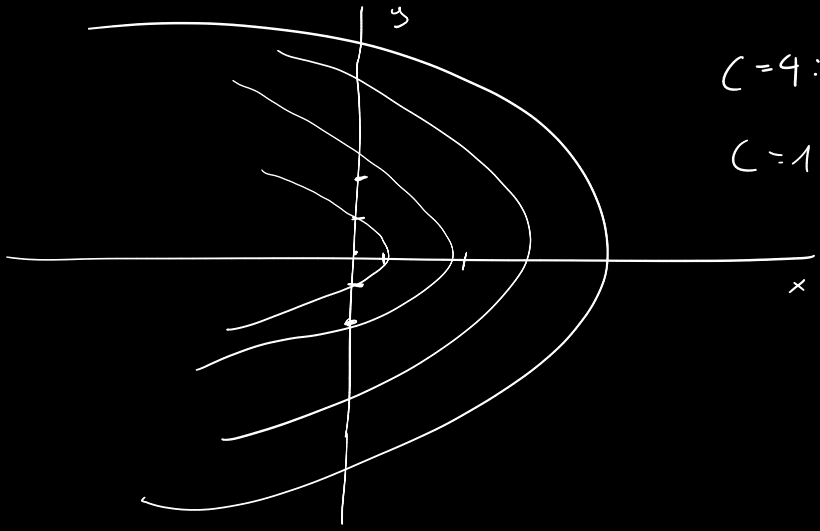
$$\frac{dy}{dx} = \frac{-1}{y}$$

$$\int y\,dy = \int -1\,dx$$

$$\frac{y^2}{2} = -x + C$$

$$x = \frac{C - y^2}{2}$$

rešitev!

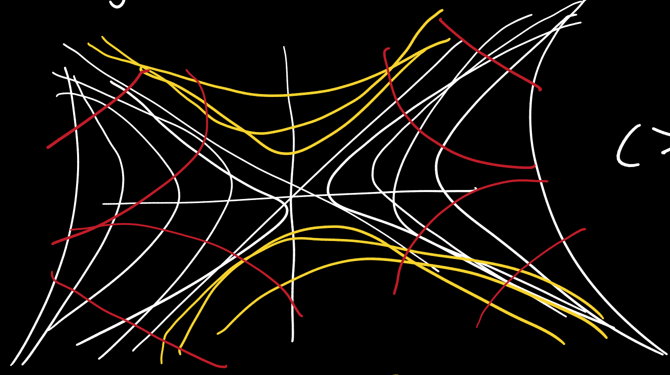


$C=4: \quad x = \frac{4-y^2}{2}$

$C=1: \quad x = \frac{1-y^2}{2}$

ŠE EN KRATEK PRIMER:

Družina: $x^2 - y^2 = C$ Določi ort. kriv.



$C>0$

$C<0$

$$x^2 - y^2 = C$$

$$\Big\downarrow \frac{d}{dx}$$

$$\cancel{2}x - \cancel{2}yy' = 0$$

$$x - yy' = 0$$

$$y' = \frac{x}{y}$$

ORTOG. KRIV:

$$y' = -\frac{y}{x} \Rightarrow \frac{y'}{y} = -\frac{1}{x} \Rightarrow \int \frac{dy}{y} = \int -\frac{dx}{x}$$

$$\ln|y| = -\ln|x| + C$$

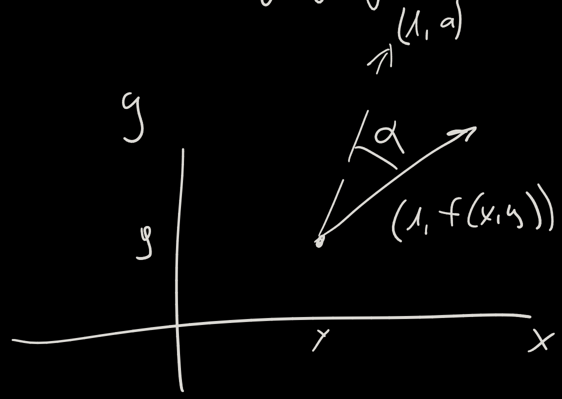
$$\ln|y| = \ln\frac{1}{|x|} + c$$

$$|y| = e^{c} \cdot \frac{1}{|x|}$$

$$\underbrace{y = \frac{D}{x}}$$ ortogonalne trajektorije

KAJ PA KOT, KI NI 90°, recimo 30°?

Pogoj je bolj kompliciran



$a = ?$

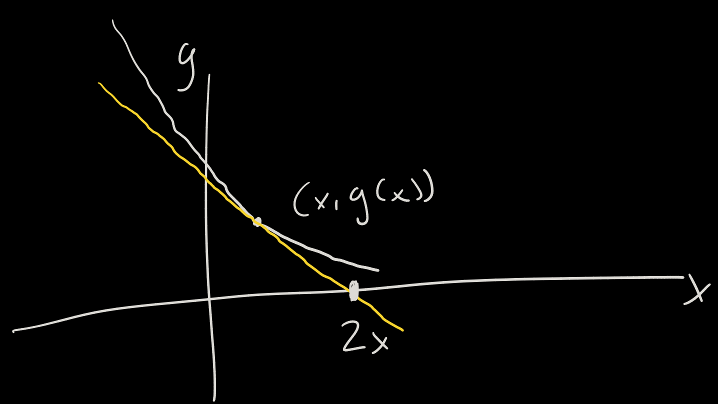
$$\cos\varphi = \frac{1+af}{\sqrt{1+f^2}\sqrt{1+a^2}}$$

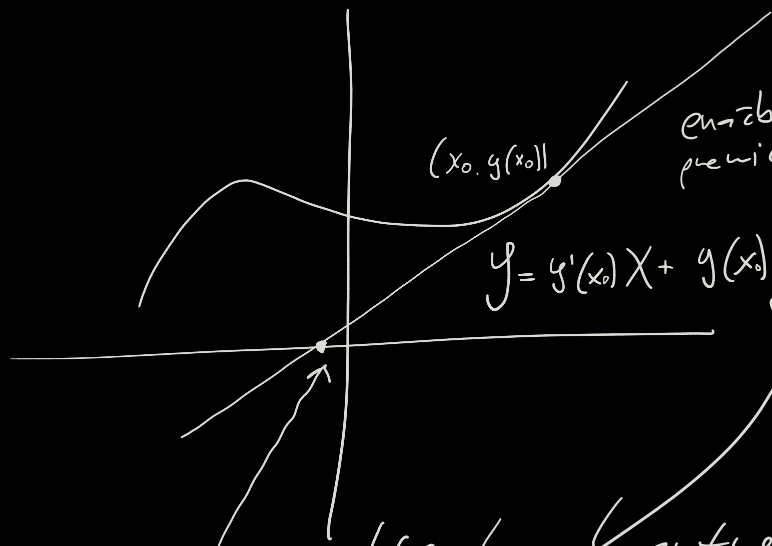
izraziti a je težko.

N

Še nekaj geometrijskih problemov:

Poišči vse fje $y(x)$ tako, da je abscisa tangente na graf fje y skozi $(x, y(x))$ enaka $2x$.





enačba tang. premice:

$$y = y'(x_0)X + y(x_0) - x_0 y'(x_0)$$

Kje tangentna premica seka x-os?

$$y = 0$$

$$y'(x_0)X + y(x_0) - x_0 y'(x_0) = 0$$

$$X = \frac{x_0 y'(x_0) - y(x_0)}{y'(x_0)}$$

Za naše navodilo vstavimo:

$$2x = \frac{x y'(x) - y(x)}{y'(x)}$$

$$2x\, y'(x) = x y'(x) - y(x)$$

$$y(x) = x y'(x) - 2x y'(x)$$

$$y = -x y'$$

$$y = -x \frac{dy}{dx}$$

$$\frac{y}{dy} = -\frac{x}{dx}$$

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y| = -\ln|x| + C$$

$$|y| = e^{\ln|x|^{-1}} e^{C}$$

$$y = \pm e^{C} e^{\ln|x|^{-1}}$$

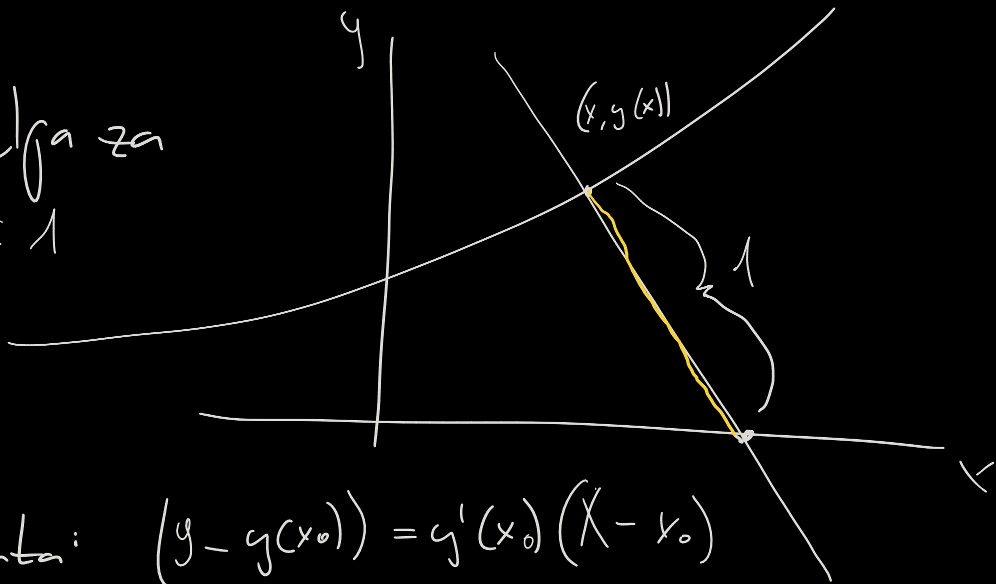
$$y = D|x|^{-1}$$

$$y = D \cdot \frac{1}{x}$$

N

Poišči fjo $y(x)$ tako, da ima odsek normale med grafom fje in x osjo konstantno dolžino 1.

Očitno velja za
$f(x) = \pm 1$



tangenta: $(y - y(x_0)) = y'(x_0)(X - x_0)$

normala: $(y - y(x_0)) = \frac{-1}{y'(x_0)}(X - x_0)$

Kje seka normala x os? vstavi $y=0$:

$$-y(x_0) = \frac{-1}{y'(x_0)}(X - x_0)$$

$$y(x_0)\,y'(x_0) = X - x_0$$

$$X = y(x_0)\,y'(x_0) + x_0$$

razdalja med ∘ in $(x_0, y(x_0))$ je:

$$\sqrt{(y(x_0)y'(x_0))^2 + y(x_0)^2} = 1$$

navodilo

$$y^2 y'^2 + y^2 = 1$$

$$y'^2 = \frac{1-y^2}{y^2}$$

$$y' = \pm\sqrt{\frac{1-y^2}{y^2}} = \pm\frac{\sqrt{1-y^2}}{y}$$

$$\frac{y'y}{\sqrt{1-y^2}} = \pm 1$$

$$\int \frac{y}{\sqrt{1-y^2}}\,dy = \pm \int 1\,dx$$

$\ln|x|$

$$1-y^2 = t$$
$$-2y\,dy = dt$$
$$dy = \frac{-dt}{2y}$$

$$\int \frac{\cancel{y}}{\sqrt{t}} \frac{-dt}{2\cancel{y}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2}\,dt =$$

$$= -\frac{\cancel{1}}{\cancel{2}} \cdot \frac{t^{1/2}}{\cancel{1/2}}$$

vstavi definicijo t

$$\cancel{-}\sqrt{1-y^2} = \pm x + C = \cancel{\pm}(x+c)^2$$

$$1-y^2 = (x+c)^2$$

$$\boxed{(x+c)^2 + y^2 = 1}$$

KROŽNICE RADIJA 1
S SREDIŠČEM V $(-c, 0)$