1. Dana je fijsta vrsta $f_x = \sum_{n=1}^{\infty} nxe^{-nx^2}$

a) Določi konv. območje fijste vrste:

$x = 0$ ✓

$x \neq 0$: konv. krit.:

$$\lim_{n\to\infty} \sqrt[n]{|nxe^{-nx^2}|} = \lim_{n\to\infty} \underbrace{\sqrt[n]{n}}_{\to 1}\,\underbrace{\sqrt[n]{x}}_{\to 1}\, e^{-x^2} < 1$$

b.) Ali fijsta vrsta konv. enak. na $\mathbb{R}$?

Ne: n-ti člen vrste:

nxe^{-nx^2} — max vrednost dosežena v $x = \pm\frac{1}{2\sqrt{n}}$

$$\sqrt{n/2}\; e^{-1} \xrightarrow{n\to\infty} \infty, \text{ torej}$$

členi vrste ne konv. enak. na $\mathbb{R}$ k $0 \Longrightarrow$

fijsta vrsta ne konv. enak. na $\mathbb{R}$.

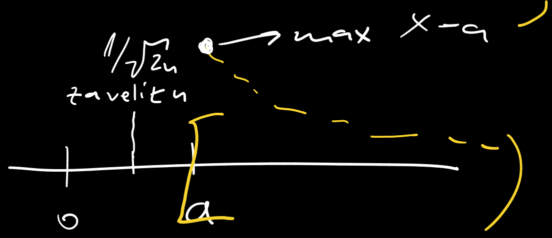
c.) Ali fijsta vrsta konv. enak. na $[a,\infty) \subset (0,\infty)$?

Uporabimo Weierstrassov kriterij

$f_n(x) = nx\, e^{-nx^2}$ $\quad \hookrightarrow \sup\limits_{x\in[0,\infty)} |f_n x|$

$$\sup_{x\in[a,\infty)} |f_n x| = \underbrace{na\, e^{-na^2}}\ \text{za dovolj velik } n$$

ker je max vrednost $f_n x$ dosežena v $x = \frac{1}{\sqrt{2n}} \xrightarrow{n\to\infty} 0$



$\sum_{n=1}^{\infty} n a e^{-na^2}$ ali konvergira? DA

$\Downarrow$ po Weierstrassu konvergira vrsta enakomerno

$\sum_{n=1}^{\infty} n x e^{-nx^2}$ na intervalu $[a,\infty)$ $a>0$.

d.) seštej to vrsto!

$f(0)=0$ f je liha fja, zadošča sešteti za $x\in\mathbb{R}^+$

$$f(x)=\sum_{n=1}^{\infty} nx\left(e^{-x^2}\right)^n$$

enak. konv.: lahko menjamo $\int$ in $\sum$

znane vsote:

$$1+x+x^2+x^4+\dots=\frac{1}{1-x} \quad \text{za } x\in(-1,1)$$

$$\int f x\,dx=\sum_{n=1}^{\infty}\int nx e^{-nx^2}\,dx=\sum_{n=1}^{\infty}\int \boxed{\tfrac{1}{2}}e^{-t}\,du=$$

$t=nx^2$

$dt=2nx\,dx$

$\frac{dt}{\boxed{2}nx}=dx$

$$=\sum_{n=1}^{\infty}-\frac{1}{2}e^{-t}=$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}e^{-nx^2}=-\frac{1}{2}\sum_{n=1}^{\infty}\left(e^{-x^2}\right)^n=$$

$$=-\frac{1}{2}\left(e^{-x^2}+\left(e^{-x^2}\right)^2+\left(e^{-x^2}\right)^3+\dots\right)=-\frac{1}{2}e^{-x^2}\left(1+e^{-x^2}+\left(e^{-x^2}\right)^2\dots\right.$$

$$= -\frac{1}{2}e^{-x^2}\frac{1}{1-e^{-x^2}} + C$$

$\downarrow$ vsota geom. vrste

↳ odvajamo, da dobimo f_x:

$$f_x = \left(\left(-\frac{1}{2}\right)\frac{e^{-x^2}}{1-e^{-x^2}} + c\right)' =$$

$$= \left(-\frac{1}{2}\right)\cdot\frac{e^{-x^2}(-2x)(1-e^{-x^2}) - e^{-x^2}\left(-e^{-x^2}(-2x)\right)}{(1-e^{-x^2})^2} =$$

$$= \left(\cancel{+}\frac{1}{\cancel{2}}\right)\cdot\frac{\cancel{+2}xe^{-x^2} \cancel{+2xe^{-2x^2} - e^{-2x^2}2x}}{(1-e^{-x^2})^2} =$$

$$= \frac{xe^{-x^2}}{(1-e^{-x^2})^2} \quad \text{za } x \neq 0$$

$$\text{za } x = 0 \quad f(0) = 0$$

[POTENČNE VRSTE IN TAYLORJEVA VRSTA]

$$\underbrace{\sum_{n=0}^{\infty} a_n (x-x_0)^n}_{} = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

↳ taki fiksni vrsti pravimo potenčna vrsta s središčem v x_0.

Kje konvergira? $x = x_0$ ✓ vsi členi so 0

v p.v. obstaja konvergenčni radij $R \geq 0$ in velja, da potenčna vrsta konvergira na $(x_0 - R, x_0 + R)$ in divergira na $\mathbb{R} \setminus [x_0 - R, x_0 + R]$. Za $x = x_0 \pm R$ pa je treba preveriti ročno.

Kako izračunati konvergenčni radij?

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

več načinov:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

kaj če je velikokrat 0?

$$\frac{1}{R} = \lim_{n\to\infty} \underbrace{\sqrt[n]{|a_n|}}_{}$$

kaj če nima stekališča

$$\frac{1}{R} = \underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}_{}$$

↳ največje stekališče $\sqrt[n]{|a_n|}$

Pri potenčnih vrstah se sme menjati $\Sigma / \int$ in $\Sigma / ()'$.

(členoma odvajamo, členoma integriramo lahko)

N Izračunaj konvergenčni radij in določi konv. obm. vrst.:

a) $$\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = \lim_{n\to\infty} \frac{n+1}{n+2} \stackrel{L.H.}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

konv.: $(-1, 1)$ div: $(-\infty, -1) \cup (1, \infty)$

kaj pa $x = 1$, $x = -1$?

$x = 1$ $$\sum_{n=0}^{\infty} (n+1)1 = 1 + 2 + 3 + 4 + \dots \text{ divergira}$$

$x = -1$ $$\sum_{n=0}^{\infty} (n+1)(-1)^n = \overbrace{1-2}^{-1} + \overbrace{3-4}^{-1} + \overbrace{5-6}^{-1} + \dots \text{ divergira}$$

konv. obm. je $(-1, 1)$.

b.) $\sum_{n=0}^{\infty} \frac{x^n}{n^n} = x + \frac{x^2}{4} + \frac{x^3}{27} + \dots$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n} = 0$$

konv radij je ∞, povsod konvergira.

c.) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n 4^n} = \frac{x^2}{4} + \frac{x^4}{32} + \frac{x^6}{192} + \dots$

$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$a_0 = 0$
$a_1 = 0$
$a_2 = 1/4$
$a_3 = 0$
$a_4 = 1/32$
$a_5 = 0$
$a_6 = 1/192$

prve formule ne moremo uporabiti.

kaj pa $\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ tudi ne.

zaporedje $\sqrt[n]{|a_n|}$ ima neskonč ∞ členov $= 0$

in ∞ členov $\neq 0$.

ni nujno, da limita obstaja,
zaporedje ima lahko več stekališč

lihi indeksi: $a_{2n+1} = 0$ sodi: $a_{2n} = \frac{1}{n4^n}$

stekališče 0

$$\lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = \text{gledamo le sode člene}$$

$$= \lim_{n\to\infty} \sqrt{\sqrt[n]{\frac{1}{n4^n}}} = \sqrt{\lim_{n\to\infty} \sqrt[n]{1/n4^n}} =$$

$$= \sqrt{\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}\, 4}} = \sqrt{1/4} = \frac{1}{2} \qquad R=2$$

konv: $(-2,2)$ div: $\mathbb{R} \setminus [-2,2]$

krajevne točke:
$x \in \{-2,2\}$:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n4^n} = \sum_{n=1}^{\infty} \frac{\cancel{4^n}}{n\cancel{4^n}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

harmonična vrsta divergira

soda potenca

$$\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n4^n} = \sum_{n=1}^{\infty} \frac{2^{2n}}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ div}$$

konv. območje je torej: $(-2,2)$

OPOMBA: naloga je moč rešiti brez računanja R:

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n4^n} \longrightarrow \text{korenski kriterij:}$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^{2n}}{n4^n}\right|} = \lim_{n\to\infty} \frac{x^2}{\sqrt[n]{n}\,4} = \boxed{\frac{x^2}{4}}$$

vrsta konvergira, ko je

$$\frac{x^2}{4} < 1 \longrightarrow x^2 \in (-2,2).$$

TAYLORJEVA VRSTA:

Naj bo $f(x)$ neskončnokrat odv. v okolici x_0.

Taylorjeva vrsta f-je f v točki x_0 je potenčna vrsta oblike:

$$\underbrace{fx \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$$

če je f analitična

Če želimo fx razviti v Taylorjevo vrsto:

- Težje: Izračunaj vse odvode
- Lažje: Uporabimo znane razvoje:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = x^2 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$x \in (-1, 1)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

$x \in (-1, 1]$

BINOMSKA VRSTA: $\alpha \in \mathbb{R}$: $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$

$x \in (-1, 1)$

za $\alpha \in \mathbb{R}$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

N

Razvij v Taylorjevo vrsto okoli 0: a) $f(x)=\frac{x}{1+x^2}=$

$$= x\cdot\frac{1}{1+x^2}$$

Znana vrsta:

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots$$

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\dots$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-x^{10}+\dots$$

$$x\left(\frac{1}{1+x^2}\right)=x-x^3+x^5-x^7+x^9-x^{11}+\dots$$

b) $f(x)=\frac{1}{(1-x)(1+2x)}=\frac{A}{1-x}+\frac{B}{1+2x}=$

$$=\frac{1/3}{1-x}+\frac{2/3}{1+2x}=\frac{1}{3}\left(\frac{1}{1-x}\right)+\frac{2}{3}\left(\frac{1}{1-(-2x)}\right)=$$

$$=\frac{1}{3}\left(1+x+x^2+\dots\right)+\frac{2}{3}\left(1-2x+4x^2-8x^3+\dots\right)=$$

$$=1\quad -x\quad +3x^2-\dots$$

N

Razvij $f(x)=\sin x$ v okolici $\frac{\pi}{3}$:

$$\sin x = \sin\left(\left(x-\frac{\pi}{3}\right)+\frac{\pi}{3}\right) = \sin\left(x-\frac{\pi}{3}\right)\underset{1/2}{\cos\frac{\pi}{3}} + \cos\left(x-\frac{\pi}{3}\right)\underset{\sqrt{3}/2}{\sin\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\left(x-\frac{\pi}{3}\right) - \frac{\left(x-\frac{\pi}{3}\right)^3}{3!} + \frac{\left(x-\frac{\pi}{3}\right)^5}{5!} + \dots\right) + \frac{\sqrt{3}}{2}\left(1 - \frac{\left(x-\frac{\pi}{3}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{3}\right)^4}{4!} + \dots\right) =$$

$$= \frac{\sqrt{3}}{2} + \frac{x-\frac{\pi}{3}}{2} - \frac{\sqrt{3}\left(x-\frac{\pi}{3}\right)^2}{4} - \dots$$

Primer težjega razvoja:

$$f(x) = \frac{\ln(1+x)}{1+x} \qquad \text{okoli } x_0 = 0$$

$$\frac{\ln(1+x)}{1+x} = \ln(1+x)\cdot\frac{1}{1+x} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots\right)\cdot$$

$$\cdot\left(1 - x + x^2 - x^3 + x^4 - x^5 \dots\right) =$$

$$= x + \left(-\frac{1}{2} - 1\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 +$$

$$+ \left(-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)x^4 + \dots$$

N

Določi konv. obm. pot. vr. in seštej!

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$x=0$ konvergira.

$$a_{2n} = 0$$

$$a_{2n+1} = \frac{(-1)^n}{2n+1}$$

$$\lim_{n\to\infty}\sqrt[2n+1]{|a_{2n+1}|} = \lim_{n\to\infty}\sqrt[2n+1]{\left|\frac{(-1)^n}{2n+1}\right|} = \lim_{n\to\infty}\frac{1}{\underset{\to 1}{\sqrt[2n+1]{2n+1}}} = 1$$

$R=1$ konv: $(-1,1)$, div $\mathbb{R}\setminus[-1,1]$

$x\in\{-1,1\}$:

$x=1$ $\quad \sum_{n=0}^{\infty}\frac{(-1)^n\, 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\dots$

$x=-1$ $\quad \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1} =$

$= -1+\frac{1}{3}-\frac{1}{5}+\dots$

konvergira.

Leibnizov kriterij

- členi proti 0
- strogo monotoni absolutno padajo $\downarrow 0$ $\Rightarrow$ konvergira

konv. obm.: $[-1,1]$.

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$ seštej!

$$f'(x)=\sum_{n=0}^{\infty}(-1)^n x^{2n} = 1-x^2+x^4-x^6\dots = \frac{1}{1+x^2}$$

$$f(x)=\int f'(x)\,dx = \int\frac{1}{1+x^2}\,dx = \arctan x + C$$

$C=?$ $\quad f(0)=0$

$\operatorname{atan} 0 = 0$

$\Rightarrow C=0$

$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots$

$f(x)=\operatorname{atan} x$

$\hookrightarrow$ $x\in(-1,1)$, zvezni sta, zato $x\in[-1,1]$.

Uporaba Taylorjevih vrst za računanje limit:

$$\lim_{x \to 0} \frac{e^{x^2}-1}{1-\cos x} = \text{lahko rešimo z L.H.}$$

tokrat pa s Taylorjevimi vrstami:

$$= \lim_{x \to 0} \frac{1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots - 1}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)} =$$

$$= \lim_{x \to 0} \frac{x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \dots}{\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} - \dots} \quad \begin{matrix} : x^2 \\ : x^2 \end{matrix} =$$

$$= \lim_{x \to 0} \frac{1 + \frac{x^2}{2} + \frac{x^4}{6} + \dots}{\frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{720} + \dots} = \frac{1}{1/2} = 2$$

D.N.

$$\lim_{x \to 0} \frac{\sqrt[3]{1+6x^2} - e^{2x^2}}{x^4}$$

$$(1+6x^2)^{1/3} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} (6x^2)^n$$

...

Diferencialne enačbe 1. reda

iščemo fjo y: $\underbrace{y'(x) = f(x, y(x))}_{\text{kvalitativni del}}$ $\underbrace{\left(+ \text{ začetni pogoj } y(x_0) = y_0\right)}_{\text{kvantitativni del}}$

primer: $y' = y$: e^x je rešitev.

TIPI:

1. Dif. en. z ločljivimi spremenljivkami:

$$y' = \frac{f(x)}{g(y)} \Longrightarrow g(y)\, y' = fx$$

$$= g(y)\, \frac{dy}{dx} = fx$$

$$gy\, dy = fx\, dx$$

$$\int gy\, dy = fx\, dx$$

$$Gy = Fx + C$$

N~

Poišči rešitev dife $y^2 = xyy' + y$ z: $y(1) = 2$ (*)

$$\frac{y^2 - y}{xy} = y'$$

$$\frac{y^2 - y}{x} = y'y$$

$$\frac{1}{x} = \frac{y'y}{y^2 - y}$$

$$\frac{1}{x} = \frac{dy\, y}{(y^2 - y)\, dx}$$

$$\frac{1}{x}\, dx = \frac{y}{y^2 - y}\, dy$$

$$\ln|x| = \int \frac{1}{x}\, dx = \int \frac{y}{y^2 - y}\, dy =$$

$$y^2 - 9 = t$$
$$2y\,dy = dt$$
$$dy = \frac{dt}{2y}$$

$$= \int \frac{\cancel{y}}{t} \cdot \frac{dt}{2\cancel{y}} = \frac{1}{2}\int \frac{1}{t}\,dt =$$
$$= \frac{1}{2} \ln|t| + C =$$
$$= \frac{1}{2} \ln|y^2 - 9|$$

$$\ln|x| + C = \frac{1}{2}\ln|y^2 - 9|$$

$$C = \frac{1}{2}\ln|y^2 - 9| - \ln|x|$$
$$C = \frac{1}{2}\ln|-5| - \ln|1| = \frac{1}{2}\ln 5 - \cancel{\ln 1}^{\,0} = \frac{1}{2}\ln 5$$

$$\Downarrow$$

$$\frac{1}{2}\ln|y^2 + 9| = \ln|x| + \frac{1}{2}\ln 5 \quad / \cdot 2$$
$$\ln|y^2 + 9| = 2\ln|x| + \ln 5 \quad / \exp$$
$$|y^2 + 9| = e^{2\ln|x| + \ln 5} = e^{\ln|x|^2} \cdot e^{\ln 5} =$$
$$= x^2 \cdot 5$$

$$|y^2 - 9| = 5x^2$$
$$y^2 - 9 = \pm 5x^2$$
$$y^2 = \pm 5x^2 + 9$$
$$y = \pm\sqrt{\pm 5x^2 + 9}$$
pogoj (*)

$$y = -\sqrt{\pm 5x^2 + 9}$$
da bo +

$$y = -\sqrt{9 - 5x^2}$$

2. TIP: Homogena diferencialna enačba:

$$y' = f\left(\frac{y}{x}\right)$$

↳ uvedemo novo fjo

$$z(x) = \frac{y(x)}{x} \quad \text{oz} \quad z = \frac{y}{x}$$

$$\Downarrow$$

$$y = x \cdot z$$

$$\Downarrow$$

$$y' = z + xz'$$

$$z + xz' = f(z)$$

$$xz' = f(z) - z$$

$$\frac{z'}{f(z) - z} = \frac{1}{x} \quad z' = \frac{dz}{dx} \Longrightarrow \int \frac{1}{f(z)-z}\, dz = \int \frac{1}{x}\, dx \Longrightarrow z(x, c)$$

$$y = x \cdot z(x, c)$$

N

Poišči rešitev: $x \cdot y' - y = x \cdot tg\left(\frac{y}{x}\right)$

da velja $y(3) = 8\pi$

$z = \frac{y}{x}$

$y = x \cdot z$

$$y' = \frac{x\, tg \frac{y}{x} + y}{x} \longrightarrow y' = tg\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)$$

$y' = z + xz'$

$$\cancel{z} + xz' = tg\left(\frac{\cancel{x}z}{\cancel{x}}\right) + \frac{\cancel{x}z}{\cancel{x}} =$$

$$= tg\, z \cancel{+ z}$$

$$xz' = tg z$$

$$x \frac{dz}{dx} = tg z$$

$$\frac{x}{dx} = \frac{tg z}{dz}$$

$$\ln|x| + C = \int \frac{1}{x} dx = \int \frac{1}{tg z} dz = \int \frac{\cos z}{\sin z} dz = \int \frac{\cancel{\cos z}}{u} \frac{du}{\cancel{\cos z}} =$$

$$u = \sin z$$

$$du = \cos z \, dz$$

$$\frac{du}{\cos z} = dz$$

$$= \int \frac{1}{u} du = \ln|\sin z|$$

$$\ln|\sin z| = \ln|x| + C$$