KRIVULJE V $\mathbb{R}^2$

Pot: $\vec{r} : [a,b] \to \mathbb{R}^3$
$\quad\quad\;\; \psi$
$\quad\quad\;\; t$

odvod

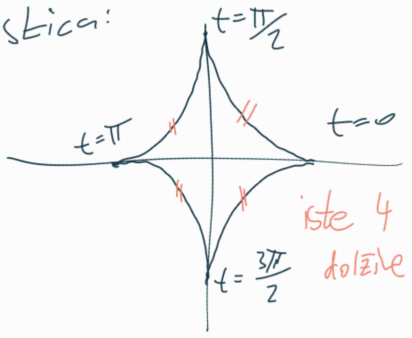
$\vec{r}(t) = (x(t), y(t), z(t))$, $\quad \dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

dolžina poti: $L = \int_a^b |\dot{\vec{r}}(t) dt|$

če je $\vec{r}$ injektivna, je dolžina poti = dolžini poti, ki jo $\vec{r}$ parametrizira.

N izračunaj dolžino krivulje astroide, parametrizirane
z $\vec{r}(t) = (a\cos^3 t, a\sin^3 t)$, $a > 0$, $t \in [0, 2\pi)$

skica:



$\dot{\vec{r}} = (3a\cos^2 t \cdot (-\sin t), 3a\sin^2 t \cos t)$

$L = \int_0^{2\pi} \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\, dt =$

$= \int_0^{2\pi} \sqrt{9a^2 \cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{1}}\, dt = \int_0^{2\pi} 3a|\cos t \sin t|\, dt =$

$= 3a \int_0^{2\pi} |\cos t \sin t|\, dt = 4 \cdot 3a \int_0^{\frac{\pi}{2}} \underbrace{\cos t \sin t}_{\text{pozitivna na } [0, \frac{\pi}{2}]}\, dt = 12a \int_0^1 \cancel{\cos t}\, u \frac{du}{\cancel{\cos t}} =$

$u = \sin t$
$du = \cos t\, dt$
$dt = \frac{du}{\cos t}$

$= 12a \int_0^1 u\, du =$

$= 12a \cdot \frac{u^2}{2}\Big|_0^1 =$

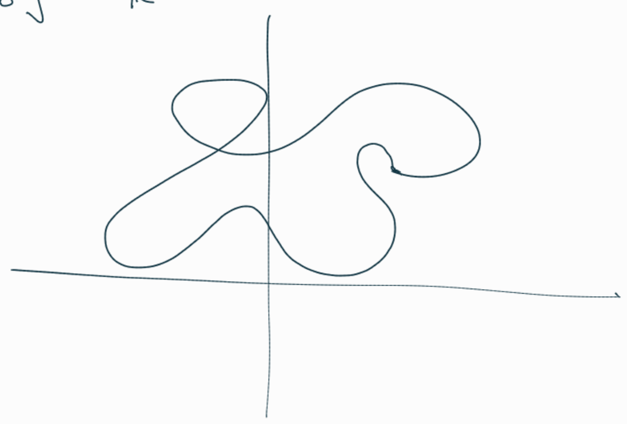
$= 12a\left(\frac{1}{2} - \frac{0}{2}\right) = \underline{\underline{6a}}$

Kaj pa ploščina?

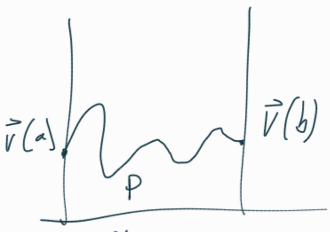
(i.) $f : [a,b] \to \mathbb{R}$ zvezna

$P = \int_a^b f(x)\,dx$.

(ii.) $\vec{r} : [a,b] \to \mathbb{R}^2$



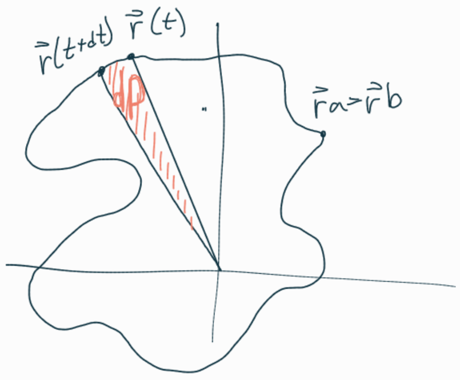
$\vec{r}(t) = (xt, yt)$
$t \in [a,b]$
Recimo $\dot{x}t \geq 0$



$P = \int_a^b y t\, \dot{x} t\, dt$

III. $\vec r(t)$ $t\in[a,b]$

parametrizira enost. skl. kriv.



orientacija naj bo pozitivna, s krivuljo omejeno območje naj bo vedno na levi, to se sprehajamo v naraščajoči smeri t po tiru.

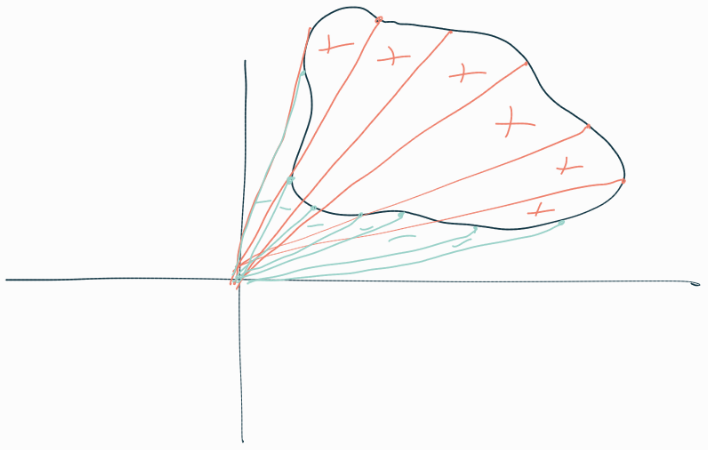
$$dP=\frac12\left|\vec r(t)\times\vec r(t+dt)\right|=\frac12\left|(x_t,y_t,0)\times(x_t+\dot x_t dt,\ y_t+\dot y_t dt,0)\right|=$$

$$=\frac12\left|(0,0,x_t\dot y_t dt-\dot x_t y_t dt)\right|=$$

$$=\frac12\left|(0,0,x_t\dot y_t-\dot x_t y_t)\right|dt=$$

$$=dP=\frac12(0,0,x_t\dot y_t-\dot x_t y_t)\,dt \quad /\int$$

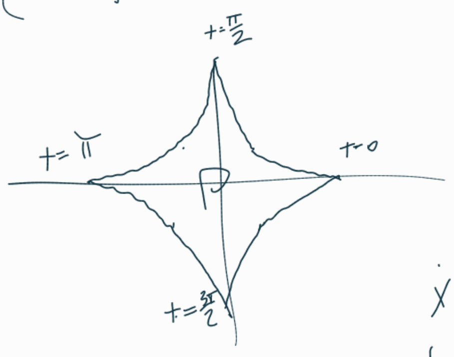
$$P=\frac12\int_a^b(x_t\dot y_t-\dot x_t y_t)\,dt$$



N

Izračunaj ploščino območja, ki ga omejuje krivulja

$\vec r(t)=(a\cos^3 t,\ a\sin^3 t)$ $a>0$ $t\in[0,2\pi)$



$$P=\frac12\int_0^{2\pi}(x_t\dot y_t-\dot x_t y_t)\,dt$$

$\dot x_t=-3a\cos^2 t\sin t$

$\dot y_t=3a\sin^2 t\cos t$

$$\cos^2 t=\frac{1+\cos 2t}{2}$$
$$\sin^2 t=\frac{1-\cos 2t}{2}$$

$$P=\frac12\int_0^{2\pi}\left(3a^2\cos^2 t\sin^4 t+3a^2\sin^4 t\cos^2 t\right)dt=$$

$$=\frac32 a^2\int_0^{2\pi}\underbrace{\cancel{(\cos^2 t+\sin^2 t)}}_{1}(\sin^2 t+\cos^2 t)$$

$$=\frac32 a^2\int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt=$$

$$=\frac32 a^2\int_0^{2\pi}\frac{(1+\cos 2t)(1-\cos 2t)}{4}\,dt=$$

$$=\frac{3a^2}{8}\int_0^{2\pi}\left(1-\cos^2(2t)\right)dt=\frac{3a^2}{8}\int_0^{2\pi}\left(1-\frac{1+\cos 4t}{2}\right)dt=$$

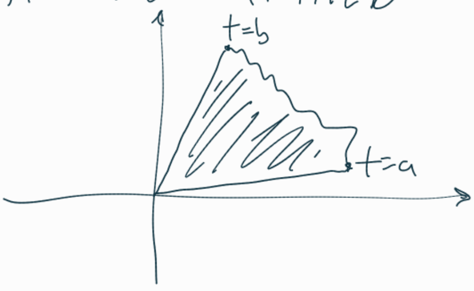
$$= \frac{3a^2}{16}\int_0^{2\pi}(1-\cos 4t)\,dt = \frac{3a^2}{16}\left(\int_0^{2\pi}1\,dt - \int_0^{2\pi}\cos 4t\,dt\right) =$$

$$= \frac{3a^2}{16}\left(2\pi - \frac{\sin 4t}{4}\Big|_0^{2\pi}\right) = \frac{3a^2}{16}(2\pi - 0) = \frac{3a^2\pi}{8}$$

pri samopresečnih krivuljah:

in $\frac{1}{2}\int_a^b \dots = P1 - P1O - P1O$

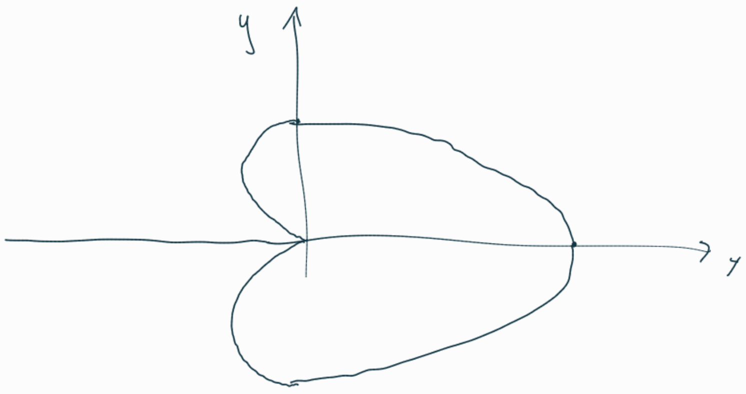
KAJ PA TAKE PRIMER:



$\vec{r}(t)$: $t \in [a,b]$

$\frac{1}{2}\int_a^b (x\dot{y} - y\dot{x})(t)\,dt$ je ploščina tega „izseka"!

N: Izračunaj ploščino lika, ki ga omejuje polarno podana krivulja z $r(\varphi) = a(1+\cos\varphi)$ $a>0$ $\varphi \in [0, 2\pi)$



Ploščina polarno parametrizirane krivulje:

$$P = \frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,d\varphi$$

Pa dajmo:

$$P = \frac{1}{2}\int_0^{2\pi}(a(1+\cos\varphi))^2\,d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi}(a + a\cos\varphi)^2\,d\varphi = \frac{1}{2}\int_0^{2\pi}(a^2 + 2a^2\cos\varphi + a^2\cos^2\varphi)\,d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi} a^2(1 + 2\cos\varphi + \cos^2\varphi)\,d\varphi =$$

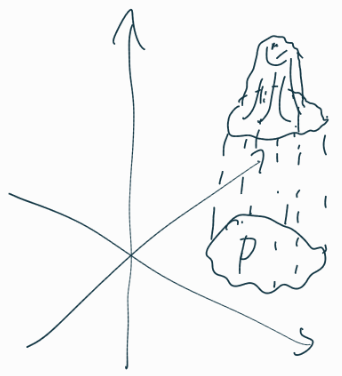
$$= \frac{a^2}{2}\left(\int_0^{2\pi}1\,d\varphi + \cancelto{0}{\int_0^{2\pi}2\cos\varphi\,d\varphi} + \int_0^{2\pi}\cos^2\varphi\,d\varphi\right) =$$

$$= \frac{a^2}{2}\left(2\pi + \int_0^{2\pi}\frac{1+\cos 2\varphi}{2}\,d\varphi\right) = \frac{a^2}{2}\left(2\pi + \frac{1}{2}\int_0^{2\pi}(1+\cos\varphi)\,d\varphi\right) =$$

$$= \frac{a^2}{2}\left(2\pi + \frac{1}{2}2\pi + \cancelto{0}{\int_0^{2\pi}\cos\varphi\,d\varphi}\right) = \pi a^2 + \frac{\pi a^2}{2} = \frac{3\pi a^2}{2}$$

PLOSKVE

eksplicitno: $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$



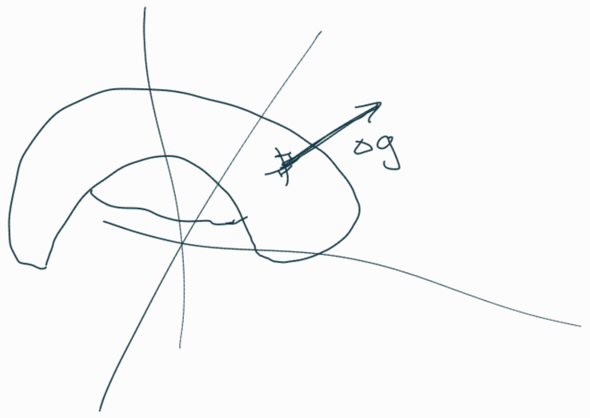
implicitno: $f: \mathbb{R}^3 \to \mathbb{R}$, zv. parc. odv. po vseh spr.

$P = \{(x,y,z) ; f(x,y,z) = 0\} \subseteq \mathbb{R}^3$

če $\underbrace{\nabla f(x,y,z) \neq 0}$ $\forall (x,y,z) \in P \Rightarrow P$ ploskev

izv. o implicitni fji $\Rightarrow$ P lahko lokalno zapišemo kot graf.

Vemo: $(x,y,z) \in P \Rightarrow \nabla f(x,y,z) \perp P$



N

Pokaži je $P = \{(x,y,z) \in \mathbb{R}^3 ; \ x^2 + y^2 - z^2 + 1 = 0\}$

Preveri, da je P ploskev in določi enačbo tangentne ravnine na P v točki $(1,1,\sqrt{3})$.

$f(x,y,z) = x^2 + y^2 - z^2 + 1$

$f_x(\dots) = 2x$

$f_y(\dots) = 2y$

$f_z(\dots) = -2z$

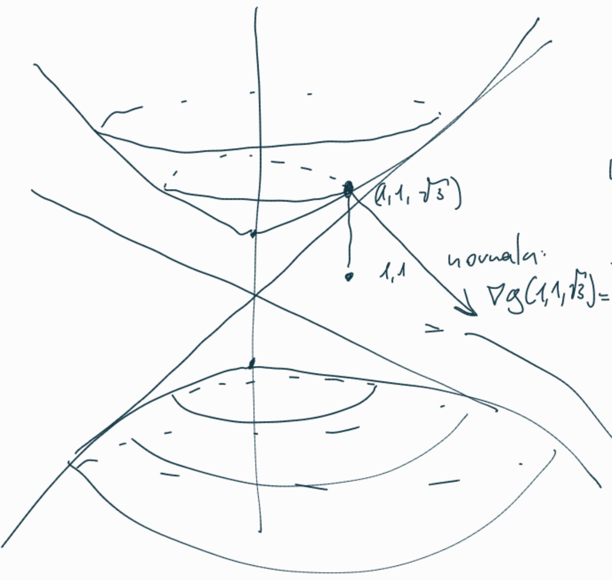
$\nabla f(x,y,z) = (2x, 2y, -2z)$

kdaj je ∇f sploh 0?

$\Leftrightarrow$ v $(0,0,0)$.

a je $(0,0,0) \in P$?

ne, ker $1 \neq 0$.



torej je P ploskev, ker $\forall a \in P: \nabla f(a) \neq 0$

P je „dvodelni hiperboloid"

$\nabla g(1,1,\sqrt{3}) = (2,2,-2\sqrt{3}) = \vec{n}$

enačba tang. ravn.: $2x + 2y - 2\sqrt{3}z = (1,1,\sqrt{3}) \cdot (2,2,-2\sqrt{3}) = -2$

N

$P = \{ (x,y,z) \in \mathbb{R}^3;\ \sin x = \operatorname{sh} y \cdot \operatorname{sh} z \}$

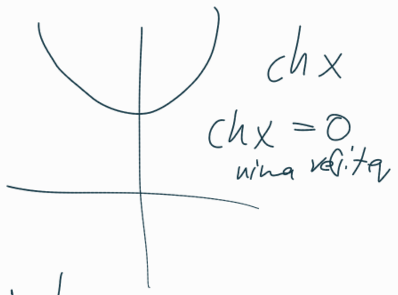
$$f(x) = \operatorname{sh} y \operatorname{sh} z - \sin x$$

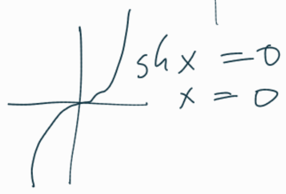
$$\nabla f x = (-\cos x,\ \operatorname{sh} z \operatorname{ch} y,\ \operatorname{sh} y \operatorname{ch} z) \overset{?}{=} (0,0,0)$$

$$x = \pi/2 + \pi k \quad k \in \mathbb{Z}$$

$$z = y = 0$$

$\nabla f x = 0 \iff$





a točke $(\frac{\pi}{2} + \pi k, 0, 0)$ $k \in \mathbb{Z}$ ustrezajo enačbi $fa = 0$?

ne.

$$f(x,y,z) = \underbrace{\operatorname{sh} y \operatorname{sh} z}_{0} - \sin x$$

$$-\sin x = \pm 1 \neq 0$$

$\forall a \in P$ velja $\nabla f(a) \neq 0 \implies P$ je ploskev.

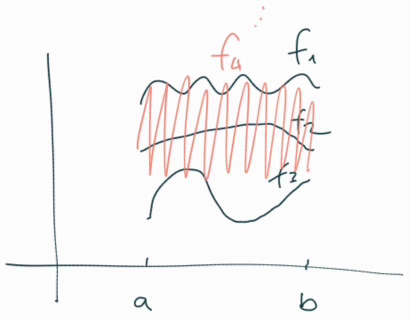
skica ploskve:

N.T.S. navedi 3d model tega!

[FUNKCIJSKA ZAPOREDJA]

funkcijsko zaporedje je zaporedje funkcij:

$$(f_n : I \to \mathbb{R})_{n \in \mathbb{N}}$$



Ali konvergira?
kam konvergira?

Def.: konv. po tč.:
$(f_n : I \to \mathbb{R})_{n \in \mathbb{N}}$ po točkah konv. k $f : I \to \mathbb{R}$, če
$\forall x \in I : \lim_{n \to \infty} f_n x = f x$

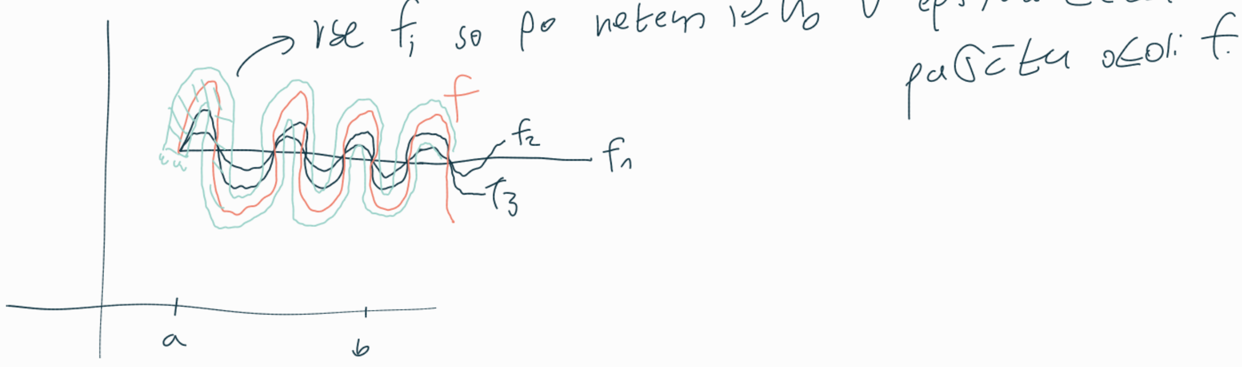
Enakomerna konvergenca:

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N} \ni: \forall n \geq n_0\ \forall x \in I : |f_n x - f x| < \varepsilon$$

$$\iff$$

$$\lim_{n \to \infty} \underbrace{\sup_{x \in I} |f_n x - f x|}_{\text{največja razlika}} = 0$$

$$\iff$$

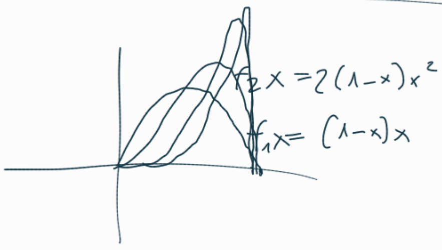


Velja:

1. $(f_n)_n$ konv. enak na I k f $\Rightarrow$ $(f_n)_n$ konv. po toč. na I k f
2. $(f_n)_n$ konv. enak na I k f in $\forall i \in \mathbb{N}: f_i$ zvezna $\Rightarrow$ f zvezna

N: dano je fijsko zaporedje:

$$f_n : [0,1] \to \mathbb{R}$$
$$f_n x = n(1-x)x^n$$



h kateri fji $f: [0,1] \to \mathbb{R}$ $(f_n)_n$ konvergira po točkah?

$\forall x \in [0,1]: f(x) = \lim_{n\to\infty} f_n x$

$x=0$ $f(0)=0$

$x=1$ $f(1)=1$

$x \in (0,1)$:

$$\lim_{n\to\infty}\left(\underset{\to\infty}{n}\,\underset{\text{konst}}{(1-x)}\,\underset{\to 0}{x^n}\right) =$$

$$(1-x)\lim_{n\to\infty}\left(n x^n\right) = (1-x)\lim_{n\to\infty}\left(\frac{\overset{\to\infty}{n}}{\underset{\infty}{\left(\frac{1}{x}\right)^n}}\right) \overset{L.H.}{=} (1-x)\lim_{n\to\infty}\frac{1}{\left(\frac{1}{x}\right)^n \ln\frac{1}{x}} =$$

$$= \frac{1-x}{\ln\frac{1}{x}}\lim_{n\to\infty}\frac{1}{\left(\frac{1}{x}\right)^n} = \frac{1-x}{\ln\frac{1}{x}}\lim_{n\to\infty} x^n = 0$$

$$\frac{d(a^x)}{dx} = a^x \ln a$$

torej $f(x) = 0$

kaj pa enakomerno? za ugotavljanje en. konv.:

$$\sup_{x\in[0,1]} \left| f_n(x) - f(x) \right|$$

iščemo max tele fje na $[0,1]$: $f_n(x) - \overset{\to 0}{f(x)} = f_n(x) = n(1-x)x^n$

odvod po x: $\left(n(1-x)x^n\right)' = n\left(-x^n + (1-x)n x^{n-1}\right) = n\left(-x^n + n x^{n-1} - n x^n\right) =$

$$= n x^n\left(-1 + n x^{-1} - n\right)$$

$$n x^n\left(-1 + n x^{-1} - n\right) = 0$$

$x = 0$

ali

$$-1 + n x^{-1} - n = 0$$
$$n x^{-1} = n+1$$
$$x^{-1} = \frac{n+1}{n}$$
$$\left[x = \frac{n}{n+1}\right]$$

$$f_n\left(\frac{n}{n+1}\right) = n\left(1 - \frac{n}{n+1}\right)\left(\frac{n}{n+1}\right)^n =$$

$$= n\left(\frac{1}{n+1}\right)\left(\frac{n^n}{(n+1)^n}\right) =$$

$$= \frac{n^{n+1}}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^{n+1}$$

max razdalja med f in f_n:

a je ta razdalja 0, ko $n\to\infty$?

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} = \lim_{n\to\infty}\left(\frac{(n+1)-1}{(n+1)}\right)^{n+1} = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^{n+1} = e^{-1} \neq 0$$

konvergenca ni enakomerna.

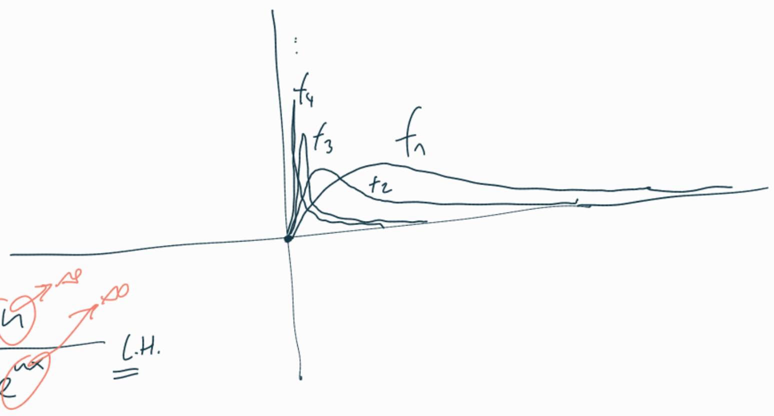
N

$(f_n : [0,\infty) \to \mathbb{R})_{n\in\mathbb{N}}$ $\quad f_n x = nxe^{-nx}$ $\quad$ a.) h kateri $f:[0,\infty)\to\mathbb{R}$ $(f_n)_n$ po točkah konvergira?

b.) ali $(f_n)_n$ na I k f konv. enakom.?

c.) let $a>0$. ali $(f_n)_n$ enak. konv. na $[a,\infty)$ k f?

a.) $f_1 x = xe^{-x}$



$fx = \lim_{n\to\infty} f_n x \qquad f0=0$

$x>0$:

$$fx = \lim_{n\to\infty} nxe^{-nx} = x\lim_{n\to\infty} \frac{n}{e^{nx}} \overset{L.H.}{=}$$

odvod po n

$$= x\lim_{n\to\infty} \frac{1}{xe^{nx}} = \lim_{n\to\infty} \frac{1}{e^{nx}} = 0$$

$(f_n)_n$ po tč konv. k $f(x)=0$

b.) $$\sup_{x\in[0,\infty)} |\underbrace{f_n x}_{\text{pozitiven}} - \underbrace{fx}_{0}| = \sup_{x\in[0,\infty)} f_n x = \sup_{x\in[0,\infty)} nxe^{-nx}$$

→ po x

kandidati za ekstrem: $(nxe^{-nx})' = 0$

vedno pozitivno

$$(nxe^{-nx})' = n(xe^{-nx})' = n(e^{-nx} - nxe^{-nx}) = n\cancel{e^{-nx}}(1-nx) =$$

$$= n - n^2x = n(1-nx)$$

$n=0$

$nx=1$

$x=\frac{1}{n}$

$$f_n\left(\frac{1}{n}\right) = \cancel{n\cdot\frac{1}{n}}\cdot e^{-n\frac{1}{n}} = e^{-1}\neq 0$$

torej tudi $\lim_{n\to\infty} e^{-1} = e^{-1} \neq 0$

torej ne konvergira enakomerno k f.

c.) naslednjič!