IZREK O IMPLICITNI fji:

(i) $D^{odp} \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ zvezno parc. odv.

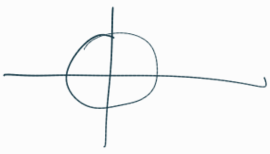
let $(a,b) \in D$, $f(a,b) = 0$ in $f_y(a,b) \neq 0$

tedaj $\exists$ okolica U točke a in $\exists!$ zv. odv $h: U \to \mathbb{R}$ $\exists$:

$h(a) = b$ in $f(x, h(x)) = 0$ $\forall x \in U$.

zadušč. $f: D \to \mathbb{R}$; $f(a,b) = 0$, ampak $f_y(a,b) = 0$. a to pomeni da $\nexists$ lok. def. h, da $h(a) = b$ in $f(x, h(x)) = 0$ $\forall x$ blizu a.

(iii) $f(x,y) = (x^2+y^2-1)^2$



$t = \{ (x,y) \in \mathbb{R}^2 ; f(x,y) = 0$

$f_x = 2(x^2+y^2-1) \cdot 2x \xrightarrow{(x,y) \in t} 0$

$f_y = 2(x^2+y^2-1) 2y \xrightarrow{(x,y) \in t} 0$

$\Downarrow$

ne moremo uporabiti izreka o implicitni fji

N od zadušč: imamo fjo $g(x,y,z) = x^3 + 3y^2 + 4xz^2 - 3z^2y - 1$.

ali: $g(x,y,z) = 0$ določa zatemo implicitno fjo $z(x,y)$ ✓ okolici $(1,0)$

$\cancel{1}\!+ 4z^2 - \cancel{1} = 0$

$z^2 = 0$

$z = 0$

$(1,0,0)$

veljati mora $g(1,0) = 0$.

$g_z(x,y,z) = 8xz - 6yz$

$g_z(1,0,0) = 0$

ne vemo izreka o implicitni fji ne moremo uporabiti.

$x^3 + 3y^2 + 4xz^2 - 3z^2y - 1$

$$z^2 = \frac{-x^3 - 3y^2 + 1}{4x - 3y}$$

$$z(x,y) = \pm \sqrt{\frac{-x^3 - 3y^2 + 1}{4x - 3y}}$$

$\hookrightarrow$ Kje je definirana?

$1 - x^3 - 3y^2 \geq 0$ in $4x - 3y > 0$

$1 - x^3 \geq 3y^2$

$y^2 \leq \frac{1-x^3}{3}$

$|y| \leq \sqrt{\frac{1-x^3}{3}}$

$3y < 4x$

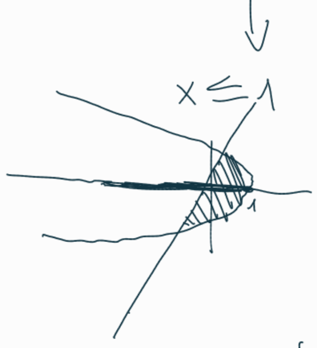
$y < \frac{4x}{3}$

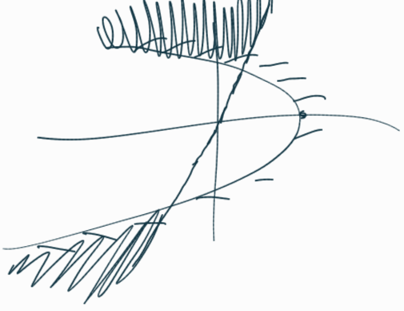
$1 - x^3 - 3y^2 \leq 0$ in $4x - 3y < 0$

$3y^2 \geq \sqrt{1-x^3}$

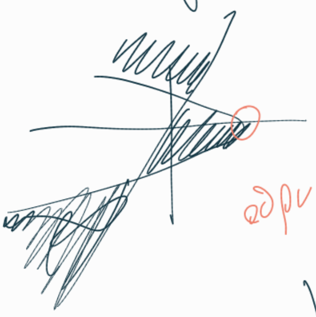
$y > \frac{4}{3}x$

$|y| \geq \sqrt{\frac{1-x^3}{3}}$





unija:



ne gre, iščemo

odprto okolico

odgovor: taka $z(x,y)$ $\nexists$

Ali enačba $x^3z^3y + xz^2y + xz + 1 = 0$ implicitno določa tako fjo $z(x,y)$, def v ok. tč. $(1,1)$? nato izr. še $z_x(1,1)$ in $z_y(1,1)$.

$$z^3 + z^2 + z + 1 = 0$$

$$z^2(z+1) + z + 1 = 0$$

$$\underbrace{(z+1)}_{0}\underbrace{(z^2+1)}_{0} = 0$$

bodisi 0 bodisi 0 (nikdar)

$\hookrightarrow z + 1 = 0$

$z = -1$

$(1,1,-1)$

$z(1,1) = -1$

$f_z(x,y,z) = 3z^2x^3y + 2xyz + x$

$f_z(1,1,-1) = 3 - 2 + 1 = 2$

lahko uporabimo izrek o implicitni fji in zato taka fja $\exists$.

izrek o i.f. pove, da je $z(x,y)$ zvezno parcialno odvedljiva.

$z(x,y)$ je takšna, da

$\forall x,y \in$ okolica $(1,1)$: $\underbrace{x^3z(x,y)^3y + xz(x,y)^2y + xz(x,y) + 1}_{\text{odvisna od } x,y\text{, ampak konstantno enaka } 0} = 0$

obe strani parc. odv.:

parc odv po x

$$3x^2z(x,y)^3y + y3z(x,y)^2z_x(x,y) + z(x,y)^2y + x2z(x,y)z_x(x,y)y + z(x,y) + xz_x(x,y) = 0$$

parc odv po y

$$x^3z(x,y)^3 + yx^33z(x,y)^2z_y(x,y) + xz(x,y)^2 + yx2z(x,y)z_y(x,y) + xz_y(x,y) = 0$$

iščemo $z_y(1,1)$ in $z_x(1,1)$: vstavimo $x = 1$, $y = 1$

$$-3+3z_x(1,1)+1-2z_x(1,1)-1+z_x(1,1)=0$$

$$2z_x(1,1)-3=0$$

$$2z_x(1,1)=3$$

$$z_x(1,1)=3/2$$

... isto za $z_y(1,1)=\dots=0$

SPOMNIMO SE: let $D^{odp}\subseteq\mathbb{R}^n$ in $f: D\to\mathbb{R}$, $(x_1,\dots,x_n)\mapsto f(x_1,\dots,x_n)$

tedaj $\operatorname{grad} f=\nabla f=(f_{x_1}, f_{x_2},\dots,f_{x_n})$

v posebnem $f:\mathbb{R}^2\to\mathbb{R}$ je $\nabla f=(f_x, f_y)$

... gradient nam pove smer, v kateri f najhitreje narašča.

(spomnimo se smernega odvoda:

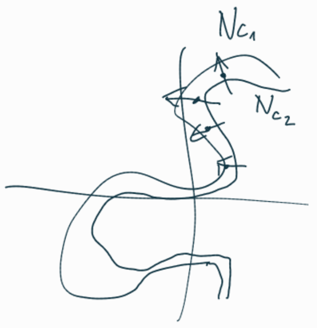
$$\frac{\partial f}{\partial \vec{s}}=\vec{s}\cdot\nabla f \quad \text{(skalarni produkt)}$$)

NIVOJNICE fje f so točke, kjer ima fja konst vrednost. (izohipse kokeheke)

$$N_c=\{(x,y)\in\mathbb{R};\ f(x,y)=c\}$$

če:

$$\forall t\in N_c: \nabla f(t)\neq 0$$



grad f bo vedno pravokoten na nivojnica

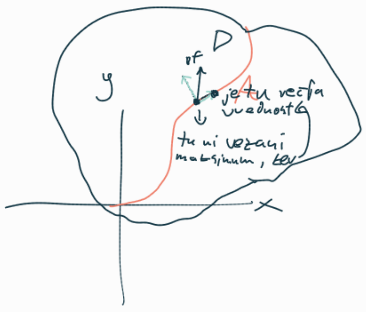
VEZANI EKSTREMI IN METODA LAGRANGEVIH MULTIPLIKATORJEV.

let $f: D^{odp\subseteq\mathbb{R}^2}\to\mathbb{R}$
$g: D\to\mathbb{R}$

$$A=\{(x,y)\in D,\ g(x,y)=0\}$$

če je v $a\in A$ dosežen lok. ekstrem f zožene na A

$$\Longrightarrow \nabla f(a)\perp A$$



$$\Updownarrow$$

$$\nabla f(a)\parallel\nabla g(a)$$

Iščemo ekstreme f zožene na A.

TRDITEV: let $f,g: D^{odp}\to\mathbb{R}$ zvezno parcialno odvedljivi. $A=\{a\in D;\ g(a)=0\}$ in let $\forall a\in A: \nabla g(a)\neq 0$. Če je v $a\in A$ dosežen vezani ekstrem fje f pri pogoju g, potem $\exists\lambda\in\mathbb{R} \ni: \nabla f(a)=\lambda\nabla g(a)$ zdb $\nabla f(a)\parallel\nabla g(a)$.

RECEPT za reševanje vezanih ekstremov:

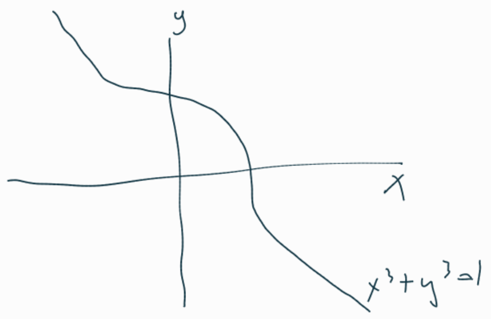
$f(x,y)$ pri pogoju $g(x,y)=0$.

$F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$. tedaj so kandidati za vezne ekstreme f pri pogoju g stac. tč. F.

$$\left.\begin{array}{l} F_x = f_x - \lambda g_x = 0 \\ F_y = f_y - \lambda g_y = 0 \end{array}\right\} \nabla f = \lambda \nabla g$$

$$F_\lambda = -\boxed{g(x,y)=0} \to \text{pogoj } g=0$$

N: Poišči VEZANE EKSTREME fje $f(x,y) = 3x^2 - 2xy + 3y^2$ pri pogoju $x^3+y^3=1 \sim g(x,y)=x^3+y^3-1=0$



$F(x,y,\lambda) = 3x^2-2xy+3y^2 - \lambda x^3 - \lambda y^3 - \lambda$

$F_x := 6x - 2y - 3\lambda x^2$

$F_y := -2x + 6y - 3\lambda y^2$

$F_\lambda := -x^3 - y^3 - 1$

$$\begin{aligned} 6x-2y-3\lambda x^2 &= 0 \\ -2x+6y-3\lambda y^2 &= 0 \\ -x^3-y^3-1 &= 0 \end{aligned}$$

$$\begin{aligned} 6x-2y &= 3\lambda x^2 \\ -2x+6y &= 3\lambda y^2 \quad /: \end{aligned}$$

$$\frac{6x-2y}{-2x+6y} = \left(\frac{x}{y}\right)^2$$

$$\frac{\cancel{y}(6\frac{x}{y}-2)}{\cancel{y}(-2\frac{x}{y}+6)} = \left(\frac{x}{y}\right)^2$$

$$t^2 = \frac{6t-2}{-2t+6} = \frac{1-3t}{t-3}$$

$$t^2(t-3) = (1-3t)$$

$$t^3 - 3t^2 - 1 + 3t$$

$$(t-1)^3 = 0$$

$$t = 1 = \frac{x}{y}$$

$$\Rightarrow \boxed{x=y}$$

$$-(x^3+y^3-1)=0$$

$$x^3+x^3-1=0$$

$$2x^3=1$$

$$x^3=\frac{1}{2}$$

$$x = \sqrt[3]{1/2} = y$$

Kandidat: $\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$

a je v $\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$ min., max. ali sedlo?

obstaja „bordered hessian matrix". out of scope

$F(x,y,\lambda) \to H(x,y,\lambda)$

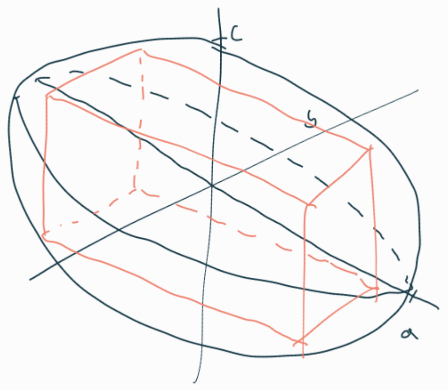
OPOMBA: konkretno za to nalogo pa lahko ugotovimo, kaj tale tč. je.

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

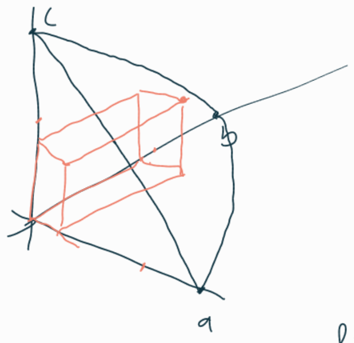
→ strogo pozitivno definitna

↳ obe lastni vrednosti sta pozitivni → od izhodišča bodo fjske vrednosti le rasle ⇒ v $\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$ je lok. min.

V troosni elipsoid včrtaj kvader največjega volumna
(polosi elipsoida in stranice kvadra so vzporedne koordinatnim osem)



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



$$V_{kvadra}(x,y,z)=8xyz$$

iščemo ekstrem

$$f(x,y,z)=\cancel{8}xyz$$

pri pogoju $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$

in $x,y,z>0$

$$F(x,y,z,\lambda)=xyz-\frac{\lambda x^2}{a^2}-\frac{\lambda y^2}{b^2}-\frac{\lambda z^2}{c^2}+\lambda$$

$$F_x(x,y,z,\lambda)=yz-\frac{2\lambda x}{a^2}$$

$$F_y(x,y,z,\lambda)=xz-\frac{2\lambda y}{b^2}$$

$$F_z(x,y,z,\lambda)=xy-\frac{2\lambda z}{c^2}$$

$$F_\lambda(x,y,z,\lambda)=-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}+1$$

$$yz-\frac{2\lambda x}{a^2}=0$$

$$xz-\frac{2\lambda y}{b^2}=0$$

$$xy-\frac{2\lambda z}{c^2}=0$$

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$$

$$y=\frac{2\lambda x}{za^2} = \frac{2\lambda x\, ab}{2\lambda\, a^2}=\frac{xb}{a}=y \qquad \frac{2\lambda \cancel{b}}{\cancel{b}ca}=y=\frac{2\lambda}{ca}$$

$$xz=\frac{(2\lambda)^2x}{za^2b^2} \qquad z=\frac{(4\lambda)^2}{za^2b^2} \qquad z^2=\left(\frac{2\lambda}{ab}\right)^2 \qquad z=\frac{2\lambda}{ab}$$

$$\frac{x^2b}{\cancel{a}}=\frac{(2\lambda)^2}{abc^2}$$

$$(xb)^2=\left(\frac{2\lambda}{c}\right)^2$$

$$xb=\frac{2\lambda}{c}$$

$$x=\frac{2\lambda}{bc}$$

vstavi x,y,z:

$$-\frac{(2\lambda)^2}{b^2c^2a^2}-\frac{(2\lambda)^2}{b^2c^2a^2}-\frac{(2\lambda)^2}{a^2b^2c^2}+1=0$$

$$3\left(\frac{2\lambda}{abc}\right)^2=1$$

$$12\left(\frac{\lambda}{abc}\right)^2=1$$

$$\lambda^2=\frac{1}{12}(abc)^2$$

$$\lambda=\frac{abc}{\sqrt{12}}=\frac{abc}{2\sqrt{3}}$$

$$x=\frac{\cancel{2}abc}{\cancel{2}\sqrt{3}\,bc}=\frac{a}{\sqrt{3}}$$

$$y=\frac{xb}{a}=\frac{\cancel{a}b}{\sqrt{3}\cancel{a}}=\frac{b}{\sqrt{3}}$$

$$z=\frac{\cancel{2}\cancel{ab}c}{\cancel{ab}\cancel{2}\sqrt{3}}=\frac{c}{\sqrt{3}}$$

alternativna rešitev:

$$yz-\frac{2\lambda x}{a^2}=0 \quad /\cdot x$$

$$xz-\frac{2\lambda y}{b^2}=0 \quad /\cdot y$$

$$xy-\frac{2\lambda z}{c^2}=0 \quad /\cdot z$$

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$$

$$\Longrightarrow \quad xyz=\frac{2\lambda x}{a^2}=\frac{2\lambda y}{b^2}=\frac{2\lambda z}{c^2}=0$$

...

Alternativen pristop k rešitvi (brez vezanega ekstrema):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$z^2 = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) c^2$$

$$z = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\; c$$

volumen je sedaj odvisen le še od x, y:

$$V(x,y) = \underbrace{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\; c\, x\, y$$

↳ omeji območje iskanja na elipso $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

izračunamo parc. odv. in stac. toč. da najdemo ekstrem $V(x,y)$

to je rešila barbara, zdaj sem pozabu vesti naslednjič.