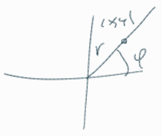
f. dveh spremenljivk.
[limite]

izračunaj: a) $\lim\limits_{(x,y)\to(1,2)} \frac{xy}{x+y} = \frac{2}{3}$

b.) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim\limits_{(x,y)\to 0,0} r\cos^2\varphi\sin\varphi$

TRIK: namesto kartezičnih (x,y) uporabi polarne koord.:

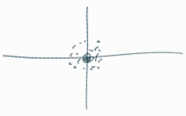


$x = r\cos\varphi$
$y = r\sin\varphi$

$$\frac{(r\cos\varphi)^2 r\sin\varphi}{(r\cos\varphi)^2 + (r\sin\varphi)^2} = \frac{r^3\cos^2\varphi\sin\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi} =$$

$$= \frac{r^{\cancel{3}}\cos^2\varphi\sin\varphi}{\cancel{r^2}(1)} = r\cos^2\varphi\sin\varphi$$

$$|r\underbrace{\cos^2\varphi\sin\varphi}_{\in[-1,1]}| \leq r$$



$$\lim_{x,y\to 0,0} \frac{x^2y}{x^2+y^2} = 0$$

$$\lim_{r\to 0} \underbrace{r}_{0}\underbrace{\boxed{\cos^2\varphi\sin\varphi}}_{\text{omejeni del}} = 0$$

$$\lim_{r\to 0} f(r\cos\varphi, r\sin\varphi) = 0 \iff \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

2. ZVEZNOST

ali je $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

zvezna na $\mathbb{R}^2$?

a je f zvezna na $\forall (x,y) \neq (0,0)$? Ja, ker je kompozitum elementarnih.

kaj pa v $(0,0)$? ne, ker $\underbrace{\lim\limits_{(x,y)\to(0,0)} f(x,y)}_{\hookrightarrow \text{ ne konvergira.}} \neq f(0,0) = 0$

gledamo funkcijske vrednosti v točkah $(x,0)$:

$$f(x) = \frac{x0}{x^2}$$

$$\lim_{x\to 0} \frac{x\cdot 0}{x^2} = 0$$

še v točkah $(0,y)$

$$f(y) = \frac{y\cdot 0}{y^2}$$

$$\lim_{y\to 0} \frac{y\cdot 0}{y^2} = 0$$

še v točkah (x,x)

$$\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2} \quad \text{ups}$$

N

$$f(x,y) \begin{cases} \frac{2x^2y}{x^4+y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{2r^2\cos^2\varphi\, r\sin\varphi}{r^4\cos^4\varphi + r^2\sin^2\varphi} = \lim_{(x,y)\to(0,0)} \frac{2r^{\cancel{3}}\cos^2\varphi\sin\varphi}{\cancel{r^2}(r^2\cos^4\varphi+\sin^2\varphi)} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{r2\cos^2\varphi\sin\varphi}{r^2\cos^4\varphi+\sin^2\varphi} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{r2\cos^2\varphi\sin\varphi}{r^2\cos^4\varphi+1-\cos^2\varphi} = \lim_{(x,y)\to(0,0)} \frac{r2\cancel{\cos^2\varphi}\sin\varphi}{\cos^2\varphi(r^2\cos^2\varphi-1)+1} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{r}2\sin\varphi}{r^{\cancel{2}}\cos^2\varphi \cancel{-1+1}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{2\sin\varphi}{r\cos^2\varphi} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{2\tan\varphi}{r\cos\varphi}$$

postavimo (x, x^2): — točke na paraboli

$$\lim_{x\to 0} \frac{2x^4}{2x^4} = 1 \qquad \text{CRT!}$$

PARCIALNI ODVODI

$$f(x) \rightsquigarrow f'(x)$$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Parcialen odvod računamo kot običajen odvod, le da drugo spr. vzamemo kot konstanto.

N

izračunaj $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$ za $f(x,y) = x^2y + x\sin(xy)$

$$\frac{\partial}{\partial x} f(x,y) = y2x + \sin(xy) + x\cos(xy)y$$

$$\frac{\partial}{\partial y} f(x,y) = x^2 + x^2\cos(xy)$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = 2y + \cos(xy)y + y\cos(xy) - xy^2\sin(xy)$$

velja $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, če sta zvezni.

na splošno o odvodu:

$$f(x_0+h) \simeq f(x_0) + f'(x_0)\cdot h$$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-f'(x_0)\cdot h}{h} = 0$$

tedaj je $f'(x_0)$ odvod v točki x_0 fje f.

odvod $f(x,y)$ v (x_0, y_0)

$$f(x_0+h, y_0+t) \approx f(x_0, y_0) + Ah + Bt.$$

odvod v (x_0, y_0) je torej $[A, B]$, če velja:

$$\lim_{(h,t)\to(0,0)} \frac{f(x_0+h, y_0+t) - f(x_0, y_0) - Ah - Bt}{\|[A, B]\|} = 0$$

definicija diferenciabilnosti:

$f(x,y)$ je diferenciabilna v (x_0, y_0), če $\exists A, B \ni$:

$$\lim_{(h,t)\to(0,0)} \frac{f(x_0+h, y_0+t) - f(x_0, y_0) - Ah - Bt}{\sqrt{h^2+t^2}} = 0$$

lastnosti diferenciabilnosti:

VELJA: f diferenciabilna v $(x_0, y_0) \Rightarrow A = \frac{\partial f}{\partial x}(x_0, y_0)$

$B = \frac{\partial f}{\partial y}(x_0, y_0)$

če je f parcialno odvedljiva v okolici (x_0, y_0) in sta $\frac{\partial f}{\partial x}$ in $\frac{\partial f}{\partial y}$ zvezni, sledi f diferenciabilna.

N

dana je $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

preveri, da je f povsod zvezna in parcialno odvedljiva. Ali je f diferenciabilna v $(0,0)$?

a.) zveznost: na ne izhodišču je elementarna.

$$\lim_{x,y\to 0,0} \frac{r\cos\varphi\, r\sin\varphi}{\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}} = \lim_{x,y\to 0,0} \frac{r^2\cos\varphi\sin\varphi}{\sqrt{r^2\cdot 1}} = \lim_{x,y\to 0,0} \underbrace{r}_{\to 0}\underbrace{\cos\varphi\sin\varphi}_{\text{omej}} = 0$$

b.) parcialno odvedljiva:

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{y\sqrt{x^2+y^2} - xy\frac{\partial}{\partial x}(x^2+y^2)\frac{1}{2\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{y\sqrt{x^2+y^2} - \frac{xy\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y\sqrt{x^2+y^2} - \frac{x^2y}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x\sqrt{y^2+x^2} - \frac{y^2x}{\sqrt{y^2+x^2}}}{y^2+x^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

ker sta parcialna odvoda zvezna, je f na diferenciabilna.

a sta $A, B = 0$ v redu za diferenciabilnost?:

$$\lim_{h,t\to 0,0} \frac{f(x_0+h, y_0+t) - f(x_0,y_0) - Ah - Bt}{\sqrt{h^2+t^2}} =$$

za $x_0, y_0 = 0,0$

$$= \lim_{h,t\to 0,0} \frac{f(h,t)}{\sqrt{h^2+t^2}} = \lim_{h,t\to 0,0} \frac{\frac{ht}{h^2+t^2}}{\sqrt{h^2+t^2}} = \lim_{h,t\to 0,0} \frac{ht}{\sqrt{h^2+t^2}\sqrt{h^2+t^2}} =$$

$$= \lim_{h,t\to 0,0} \frac{ht}{h^2+t^2}$$

po x osi: $\lim_{h\to 0} \frac{h\cdot 0}{h^2+0} = 0$

po diagonali $h=t$: $\lim_{h\to 0} \frac{h^2}{2h^2} = \frac{1}{2}$

CRE!

[redacted] torej ni diferenciabilna v $(0,0)$

POMEN DIFERENCIABILNOSTI

smerni odvod fje f v smeri vektorja

$\vec{s} = (a,b)$

oznaka: $\frac{df}{ds}(x_0,y_0) = \lim_{t\to 0} \frac{f((x_0,y_0) + t(a,b)) - f(x_0,y_0)}{t}$

$\Downarrow$ povezava med

$\frac{df}{d\vec{s}}$ in $\frac{\partial f}{\partial x}$ in $\frac{\partial f}{\partial y}$

N let f diferenciabilna v x_0, y_0 in $\vec{s} = (a,b)$

$$\frac{\partial f}{\partial \vec{s}}(x_0,y_0) = \lim_{t\to 0} \frac{f(x_0+ta, y_0+tb) - f(x_0,y_0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(x_0+ta, y_0+tb) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)ta + \frac{\partial f}{\partial x}(x_0,y_0)ta - \frac{\partial f}{\partial y}(x_0,y_0)tb + \frac{\partial f}{\partial y}(x_0,y_0)tb}{t}$$

$$= \lim_{t\to 0} \frac{f(x_0+ta, y_0+tb) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)ta - \frac{\partial f}{\partial y}(x_0,y_0)tb}{t} + \frac{\partial f}{\partial x}(\cdot)a + \frac{\partial f}{\partial y}(\cdot)b =$$

$$= \lim_{t\to 0} \frac{\sqrt{a^2t^2+b^2t^2}}{t} \cdot \frac{f(x_0+ta, y_0+tb) - f(x_0,y_0) - \frac{\partial f}{\partial x}(\dots)ta - \frac{\partial f}{\partial y}(\dots)tb}{\sqrt{a^2t^2+b^2t^2}} + \frac{\partial f}{\partial x}(\cdot)a + \frac{\partial f}{\partial y}(\cdot)b =$$

$$= a\frac{\partial f}{\partial x}(x_0,y_0) + b\frac{\partial f}{\partial y}(x_0,y_0).$$

$$\Longrightarrow \quad \frac{\partial f}{\partial \vec{s}} = a\frac{\partial f}{\partial x}(x_0,y_0) + b\frac{\partial f}{\partial y}(x_0,y_0) \quad \text{za } \vec{s} = (a,b)$$

GRADIENT $f(x_0,y_0) = \nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)$

$$\frac{\partial f}{\partial \vec{s}} = \nabla f \, \vec{s}$$

OPOMBA: Za diferenciabilne fje velja verižno pravilo:

$$\frac{df}{d\vec{s}}(x_0,y_0) = \frac{d}{dt}\Big|_{t=0} f(x_0+ta,\, y_0+tb) =$$

$$= \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b$$

zakaj rabimo smerni odvod?

$$\frac{df}{d\vec{s}} = \nabla f \vec{s}$$

2.) v kateri smeri se vrednost f ne spremeni?

če je $\vec{s}$ pravokoten na ∇f,

je $\frac{df}{d\vec{s}} = 0$

3.) v kateri smeri se vrednost f spremeni najbolj?

če je $\vec{s}$ vzporeden na ∇f.

če $\vec{s}$ kaže v isto smer kot ∇f, je $\frac{df}{d\vec{s}}$ maksimalen.

[LOKALNI EKSTREMI] f je povsod diferenciabilna.

$D \subseteq \mathbb{R}$ $\quad f: D \to \mathbb{R}$. določi točko v tem območju, kjer f doseže maksimalno vrednost

• so označeni lokalni ekstremi.

postopek iskanja:

- poiščimo kandidate za lokalne ekstreme.

kandidati so stacionarne točke, kjer je

$$\frac{\partial f}{\partial x}a = 0 = \frac{\partial f}{\partial y}a$$

dve enačbi in dve neznanki

$\Downarrow$

vse rešitve sistema so stac. točke.

- za vsako od stac. točk sedaj preverimo, ali gre za lok. min., lok. max. ali sedlo.

zapišemo hessejevo matriko:

$$H_{(x,y)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

za vsako stac. točko pogledamo matriko H_a.

če je $\det H_a < 0$, je v točki a sedlo

če je $\det H_a > 0$; $\frac{\partial^2 f}{\partial x^2} > 0 \Rightarrow$ lok. min

$\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow$ lok. max

1. Dana je f-a $f(x,y) = (x^2+y^2-3)e^x$.
določi stac. točke f in tip ekstrema ✓
vsake stac. točki.

$$\frac{\partial f}{\partial x}(x,y) = 2xe^x + (x^2+y^2-3)e^x = e^x(2x+x^2+y^2-3)$$

$$\frac{\partial f}{\partial y}(x,y) = e^x 2y$$

$$\cancel{e^x}(2x+x^2+y^2-3) = 0$$

$$\cancel{e^x 2}y = 0$$

$$x^2+2x-3=0$$

$$x_1 = 1 \qquad x_2 = -3$$

stacionarni točki sta $(1,0)$, $(-3,0)$