cont. dokaza:

če so vsi parcialni odvodi v točki a zvezni, je f diferenciabilna v točki a.

DOKAZ za $k=2$:

$$f(a+h)-f(a) = f(b+u, c+v) - f(b,c) =$$

$a \to (b,c)$, $h \to (u,v)$

$$= \big(f(b+u, c+v) - f(b+u, c)\big) + \big(f(b+u, c) - f(b,c)\big)$$

$y \mapsto f(b+u, y)$ $\qquad$ $x \mapsto f(x, c)$

Lagrangev izrek

$c^* \in (c, c+v)$ $\quad \frac{\partial f}{\partial y}(b+u, c^*)(c+v-c=v)$ $\qquad \frac{\partial f}{\partial y}(b^*, c)(u)$

$$f(a+h)-f(a) = df(a)\cdot h + R_a(h)$$

če gre ta ostanek hitreje proti 0 kot $df(a)\cdot h$

$$\lim_{h\to 0} \frac{R_a(h)}{\|h\|} = 0$$

lagrange: $f(x_1) - f(x_2) = f'(\xi)(x_2 - x_1)$

$$f(a+h)-f(a) = f(b+u, c+v) - f(b,c) = \underbrace{f(b+u, c+v) - f(b, c+v)}_{\text{lagr.}} + f(b, c+v) - f(b,c) =$$

med 0 in u $\qquad$ med 0 in v

$$\to \frac{\partial f}{\partial x}(b+s, c+v)\cdot u + \frac{\partial f}{\partial y}(b, c+t)\cdot v$$

$h = (u,v)$ $\quad a = (b,c)$

$$df(a) = \frac{\partial f}{\partial x}(b,c)\cdot u + \frac{\partial f}{\partial y}(b,c)\cdot v$$

$$R_a(h) = \frac{\partial f}{\partial x}(b+s, c+v)\cdot u + \frac{\partial f}{\partial y}(b, c+t)v - df(a) =$$

$$= \frac{\partial f}{\partial x}(b+s, c+v)\cdot u + \frac{\partial f}{\partial y}(b, c+t)\cdot v - \frac{\partial f}{\partial x}(b,c)u - \frac{\partial f}{\partial y}(b,c)\cdot v =$$

$$= \left(\frac{\partial f}{\partial x}(b+s, c+v) - \frac{\partial f}{\partial x}(b,c)\right)\cdot u + \left(\frac{\partial f}{\partial y}(b, c+t) - \frac{\partial f}{\partial y}(b,c)\right)\cdot v =$$

$$\frac{R_a(h)}{\|h\|} = \frac{(\dots)\cdot u}{\|h\|} + \frac{(\dots)\cdot v}{\|h\|}, \qquad \|h\| = \sqrt{u^2+v^2}, \qquad \frac{|u|}{\|h\|} \le 1, \quad \frac{|v|}{\|h\|} \le 1$$

$$\left|\frac{R_a(h)}{\|h\|}\right| \le \underbrace{\left|\frac{(\dots)\cdot u}{\|h\|}\right|}_{<\varepsilon} + \underbrace{\left|\frac{(\dots)\cdot v}{\|h\|}\right|}_{<\varepsilon}$$

če so parcialni odvodi zvezni, sta $(\dots) < \varepsilon$ za male $\|h\|$, saj sta s,t omejena z u,v.

$$\left|\frac{R_a(h)}{\|h\|}\right| \le 2\varepsilon \text{ za male } \|h\| \Rightarrow \lim_{h\to 0}\left|\frac{R_a(h)}{\|h\|}\right| = 0$$

za zvezne parcialne odvode.

$\square$

Def.: let $U \subset \mathbb{R}^k$ in $f: U \to \mathbb{R}$ funkcija. Če ima f v vsaki točki U prve parcialne odvode po vseh spremenljivkah in so vsi ti odvodi na U zvezni, je f diferenciabilna v vsaki točki U (po zgornjem odvodu). Rečemo, da je f zvezno odvedljiva na U in pišemo $f \in C^1(U)$.

S $C^2(U)$ označimo množico f, ki imajo na U vse parcialne odvode drugega reda, ki so zvezni vall. Imenujemo jih dvakrat zvezno odvedljive f na U.

Za $r \in \mathbb{N}$: $C^r(U)$ označuje množico f, ki imajo na U vse parcialne odvode r-tega reda, ki so zvezni.

$$\frac{\partial^r f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_r}}$$

$C^\infty(U) = \bigcap_{r\in\mathbb{N}} C^r(U)$ .. množica fj, ki imajo parcialne odvode kateregakoli reda na U. „gladke" ali „poljubnomnogokrat odvedljive fje na U".

$C^0(U)$ je množica vseh zveznih fj na U.

Izrek: enakost mešanih odvodov.

(1.) Let U odprta podmnožica v $\mathbb{R}^k$ in $f \in C^2(U)$.
Potem $\forall x \in U$ in $\forall i,j \in \{1,\dots,k\}$ velja $\frac{(\partial f)^2}{\partial x_i \partial x_j}(x) = \frac{(\partial f)^2}{\partial x_j \partial x_i}(x)$.

$\Longrightarrow$ (2.) za $f \in C^r(U)$ $r \geq 2$ vrstni red parcialnih odvodov reda r ni pomemben.

(Če velja (1.), velja (2.) $\frac{(\partial f)^2}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$ $\leftarrow \frac{\partial f}{\partial x}$).

dokazati je treba (1).:

Dokaz: $a = (b,c)$ $h = (u,v)$

$$I = f(b+u, c+v) - f(b, c+v) - f(b+u, c) - f(b,c) =$$

let $g(y) = f(b+u, y) - f(b,y)$

$g(c+v)$ zvezno odvedljiva $\qquad g(c)$

$$= g(c+v) - g(c) \overset{\text{lagrange } \exists s \text{ med } 0 \text{ in } v}{=} g'(c+s)\cdot v =$$

$$\overset{\text{def } g}{=} \left(\frac{\partial f}{\partial y}(b+u, c+s) - \frac{\partial f}{\partial y}(b, c+s)\right)\cdot v =$$

$x \longmapsto \frac{\partial f}{\partial y}(x, c+v)$ je zvezno odvedljiva

$$\overset{\text{lagrange}}{=} \frac{\partial f}{\partial x \partial y}(b+t, c+s)\cdot u \cdot v$$

$\exists t$ med 0 in u

... zavadi asociativnosti:

$$I = \big(f(b+u, c+v) - f(b, c+v)\big) - \big(f(b+u, c) - f(b,c)\big) =$$

$$= \big(f(b+u, c+v) - f(b+u, c)\big) - \big(f(b, c+v) - f(b,c)\big)$$

let $h(x) = f(x, c+v) - f(x, c)$

po lagr.:

$$I = (h(b+u) - h(b)) = \left(\frac{\partial f}{\partial x}(b+s^*, c+v) - \frac{\partial f}{\partial x}(b+s^*, c)\right)u =$$

$$= \frac{(\partial f)^2}{\partial y \partial x}(b+s^*, c+t^*)\cdot u \cdot v$$

$\Downarrow$

$$I = \frac{(\partial f)^2}{\partial x \partial y}(b+t, c+s)\cdot \cancel{u \cdot v} = ,$$ t med 0 in u, s med 0 in v

$$= \frac{(\partial f)^2}{\partial y \partial x}(b+s^*, c+t^*)\cdot \cancel{u \cdot v} = ,$$ s^* med 0 in u, t^* med 0 in v

$\hookrightarrow$ enačaj velja za vse dovolj majhne h, da so t, s, s^*, t^* blizu 0.

$$\Rightarrow \frac{(\partial f)^2}{\partial x \partial y}(b+t, c+s) = \frac{(\partial f)^2}{\partial y \partial x}(b+s^*, c+t^*)$$ za dovolj velike h in $u \neq 0$ in $v \neq 0$.

$\Rightarrow \frac{(\partial f)^2}{\partial x \partial y}(b,c) = \frac{(\partial f)^2}{\partial y \partial x}(b,c)$ (uporabimo zveznost)

↳ $f \in C^2(U)$

TAYLORJEVA FORMULA IN VERIŽNO PRAVILO.

let f diferenciabilna v $a \in \mathbb{R}^k$ in definirana v okolici a.

velja $f(a+h) \approx f(a) + df(a)\cdot h =$

$$= f(a) + \frac{\partial f}{\partial x_1}(a)\cdot h_1 + \frac{\partial f}{\partial x_2}(a)\cdot h_2 + \dots + \frac{\partial f}{\partial x_k}(a)\, h_k$$

Iščemo boljši približek s pomočjo odvodov višjega reda.

let $U \subset \mathbb{R}^k$ odprta in $f : U \to \mathbb{R}$

za dovolj majhne h daljica med a in $a+h$ leži cela v U.

$F: t \mapsto a+th$ je parametrizacija daljice $(a, a+h)$, kjer je $t \in [0,1]$. F je torej zožitev f na to 1D daljico.

$$F(t) = f(a+th) \quad \text{za } t \in [0,1].$$

$\Rightarrow F(0) = f(a)$, $F(1) = f(a+h)$

če je f $n+1$ krat zvezno diferenciabilna na U, je F $n+1$ krat zvezno odvedljiva na okolici daljice (še nedokazano)

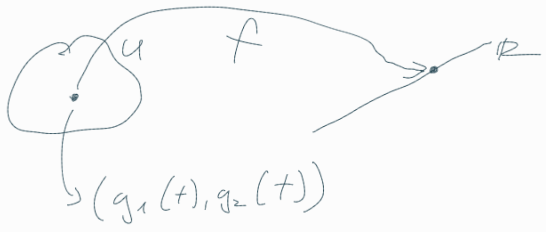
TAYLORJEVA FORMULA:

$\exists c \in (0,1)$

$$F(1) = F(0) + F'(0)\cdot 1 + \frac{F^{(2)}(0)}{2}\cdot 1^2 + \dots + \frac{F^{(n)}(0)}{n!} 1^n + \frac{F^{(n+1)}(c)}{(n+1)!} 1^{n+1}$$

IZREK (verižno pravilo): let $U \subseteq \mathbb{R}^k$ in $f: U \to \mathbb{R}$ diferenciabilna

let J odprti interval in $g_1, g_2, \dots g_k : J \to \mathbb{R}$ odvedljive f-je, za katere velja, da je $g(t) = (g_1(t), \dots, g_k(t)) \in U \ \forall t \in J$.



potem je f-ja

$F(t) = f(g(t)) =$

$= f(g_1(t), \dots, g_k(t))$

odvedljiva in njen odvod je

pravilo za kompozitumski odvod

(za $k=1$: $F'(t) = f'(g(t))\, g'(t)$)

$$F'(t) = \frac{\partial f}{\partial x_1}(g(t))\cdot g_1'(t) + \frac{\partial f}{\partial x_2}(g(t))\, g_2'(t) + \dots + \frac{\partial f}{\partial x_k}(g(t))\, g_k'(t) =$$

$$= df(g(t)) \cdot g'(t)$$

↳ skalarni produkt

↳ $(g_1'(t), g_2'(t), g_3'(t), \dots, g_k'(t))$

sedaj velja, da je F odvedljiva $n+1$ krat (zgoraj):

$$F(t) = f(a+th)$$

$$a(t) = (a_1 + th_1, \dots, a_k + th_k)$$

$$F'(t) = \frac{\partial f}{\partial x_1}(a+th)\,\underbrace{\frac{\partial(a_1+th_1)}{\partial t}} + \dots + \frac{\partial f}{\partial x_k}(a+th)\cdot\frac{\partial(a_k+th_k)}{\partial t} =$$

$$= \frac{\partial f}{\partial x_1}(a+th)\cdot h_1 + \dots + \frac{\partial f}{\partial x_k}(a+th)\,h_k$$

$$F'(0) = \frac{\partial f}{\partial x_1}(a)\,h_1 + \dots + \frac{\partial f}{\partial x_k}(a)\cdot h_k = df(a)\cdot h$$

$$F''(t) = f_{x_1x_1}(a+th)\cdot h_1^2 + f_{x_1x_2}(a+th)\,h_1h_2 + \dots + f_{x_1x_k}(a+th)\,h_kh_1 + \dots$$

$$\dots + f_{x_2x_1}(a+th)\,h_2h_1 + f_{x_2x_2}(a+th)\,h_2^2 + \dots + f_{x_2x_k}(a+th)\,h_kh_2 + \dots$$

$$\dots + \quad \dots \quad + \dots$$

$$\dots + f_{x_kx_1}(a+th)\,h_kh_1 + f_{x_kx_2}(a+th)\,h_kh_2 + \dots + f_{x_kx_k}(a+th)\,h_k^2$$

$$F''(0) = f_{x_1,x_1}(a)\,h_1^2 + \dots + f_{x_k,x_k}(a)\,h_k^2$$

$$F'(t) = \left[\left(h_1\frac{\partial}{\partial x_1} + \dots + h_k\frac{\partial}{\partial x_k}\right)f\right](a+th)$$

$$F''(x) = \left[\left(h_1\frac{\partial}{\partial x_1} + \dots + h_k\frac{\partial}{\partial x_k}\right)^2 f\right](a+th)$$

$$F^{(n)}(x) = \left[\left(h_1\frac{\partial}{\partial x_1} + \dots + h_k\frac{\partial}{\partial x_k}\right)^n f\right](a+th)$$

<u>Izrek:</u> (Taylorjeva formula)

Naj bo $U \subset \mathbb{R}^k$ odprta in $f: U \to \mathbb{R} \in C^{n+1}(U)$.
Naj bo $a \in U$ in $h \in \mathbb{R}^k$ tako majhen, da daljica $(a, a+th) \in U$. Tedaj je Taylorjev polinom stopnje n s središčem v a pri spremembi h enak

$$T_{f,a,n}(h) = \sum_{i=0}^{n} \frac{\left[\left(h_1\frac{\partial}{\partial x_1} + \dots + h_k\frac{\partial}{\partial x_k}\right)^i f\right](a)}{i!}$$

in ostanek

$$R_{f,a,n}(h) = f(a+h) - T_{f,a,n}(h) =$$

$$= \frac{\left[\left(h_1\frac{\partial}{\partial x_1} + \dots + h_k\frac{\partial}{\partial x_k}\right)^{i+1} f\right](a+th)}{(i+1)!}$$

za neki t med 0 in 1

kompaktna množica

Če je zaprta krogla $\overline{K}(a,r) \subset U$, $r > 0$,
potem $\exists M \in \mathbb{R}\ \forall h;\ \|h\| < r: \ |R_{f,a,n}(h)| \leq \frac{M}{(n+1)!}\|h\|^{n+1}$

maksimum odvodov na kompaktni množici.

<u>Opomba:</u> operator $h_1\frac{\partial}{\partial x_1} + \dots + h_k\frac{\partial}{\partial x_n} = \vec{h}\cdot\underbrace{\left(\frac{\partial}{\partial x}, \dots, \frac{\partial}{\partial x_k}\right)}_{\nabla} = h\nabla$

torej $F^{(i)}(t) = (h\nabla^i f)(a+th)$

$h \cdot \nabla f$ je smerni odvod f v smeri h.

Dokaz verižnega pravila:

Vemo: (*) $f(x+h) = f(x) + df(x)\cdot h + R_{f,x}(h)$ in

$$\lim_{h\to 0} \frac{R_{f,x}(h)}{\|h\|} = 0 \quad (f \text{ je diferenciabilna})$$

ker so g odvedljive, je (**) $g_i(t+s) = g_i'(t)\cdot s + R_{i,t}(s)$ in

velja $\lim_{s\to 0} \frac{R_i(s)}{s} = 0$ (*)

$$F(t+s) - F(t) = f(\underbrace{g(t+s)}_{x+h}) - f(\underbrace{g(t)}_{x}) =$$

$h = g(t+s) - x = g(t+s) - g(t)$

$x = g(t)$

$$\overset{(*)}{=} df(x)h + R_{f,x}(h) = \frac{\partial f}{\partial x_1}(x)h_1 + \frac{\partial f}{\partial x_2}(x)h_2 + \dots + \frac{\partial f}{\partial x_k}(x)h_k + R_{f,x}(h)$$

$$\overset{(**)}{=} \frac{\partial f}{\partial x_1}(g(t))\left(g_1'(t)\cdot s + R_{1,t}(s)\right) + \dots + \frac{\partial f}{\partial x_k}(g(t))\left(g_k'(t)\cdot s + R_{k,t}(s)\right) + R_{f,x}(h) =$$

$$= \frac{\partial f}{\partial x_1} g(t)\cdot g_1'(t)\cdot s + \dots + \frac{\partial f}{\partial x_1} g(t)\cdot g_k'(t)\cdot s + \underbrace{\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(g(t))\cdot R_{i,t}(s) + R_{f,x}(h)}_{R(s)}.$$

$$F(t+s) - F(t) = \underset{F'(t)}{dF(t)}\cdot s + R(s)$$

zaradi (*) je $\lim_{s\to 0} \frac{\Sigma}{s} = 0$

$$\lim_{s\to 0} \frac{R_{f,x}(h)}{\|h\|}\cdot\frac{\|h\|}{s}$$

$\frac{R_{f,x}(h)}{\|h\|} \to 0$, $\frac{\|h\|}{s}$ omejen

$= 0$