ANAN FMF 2023-12-01

pregled funkcij:

$f: A \to B$ funkcija.

f je injektivna, če različnim elementom priredi različne slike

f je surjektivna, če je $Z_f = D$.

N —— naj bo $a \in \mathbb{R} \setminus \mathbb{Q}$ in naj bo $f: \mathbb{N} \to \mathbb{R}$ s predpisom $f(n) = an - \lfloor an \rfloor$

oznaka: $\lfloor \check{z} \rfloor$: največje celo število, ki je manjše ali enako $\check{z}$.

dokaži, da je f injektivna in da je $Z_f \subseteq [0,1] \cap \{\mathbb{R} - \mathbb{Q}\}$

injektivnost: $f(n) = f(m)$ hočemo pretvoriti v $n = m$ $(n, m \in \mathbb{N})$

$$an - \lfloor an \rfloor = am - \lfloor am \rfloor$$

$$an - am = \lfloor an \rfloor - \lfloor am \rfloor$$

$$a(n-m) = \underbrace{\lfloor an \rfloor - \lfloor am \rfloor}_{\in \mathbb{Z}}$$

↳ mora biti 0, drugače bi lahko obe strani z njim delili in dobili $a \in \mathbb{Q}$, kar je $\times$

$$\hookrightarrow \quad a = \frac{\lfloor an \rfloor - \lfloor am \rfloor}{n-m} \quad \begin{matrix} \}\mathbb{Z} \\ \}\mathbb{Z} \end{matrix} \Big\} \mathbb{Q}$$

$\times \Rightarrow n = m$

f je inj. $\Leftarrow$

Z_f: $f(n) = an - \lfloor an \rfloor$

$an \in \mathbb{R} - \mathbb{Q}$.

$r - z = s$ $\quad r \in \mathbb{R} - \mathbb{Q}$, $z \in \mathbb{Z}$, $s \in \mathbb{R} - \mathbb{Q}$

vidimo, da je $an - \lfloor an \rfloor$ iracionalen.

po definiciji je $\lfloor r \rfloor \leq r \Rightarrow an - \lfloor an \rfloor \geq 0$

še $f(n) < 1$.

dokaz s $\times$. recimo, da je $f(n) \geq 1$.

$$an - \lfloor an \rfloor \geq 1$$

$$an \geq \lfloor an \rfloor + 1$$

trditev manjši od an $\leftarrow \lfloor an \rfloor \geq \lfloor an \rfloor + 1$

$$0 \geq 1 \quad \times$$

$\square$

N
let $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\quad x \in \mathbb{R}$.

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

a.) Pokaži, da je sinh injektivna na $\mathbb{R}$ in določi $\sinh^{-1}$.

b.) Pokaži, da je cosh injektivna na $[0,\infty)$ in določi inverz $\cosh|_{[0,\infty)}$ → zožitev na interval. [Domača naloga]

REŠITEV: $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ za $x \in [1,\infty)$

c.) Izpelji adicijske izreke za sinh in cosh.

a.) $\sinh(x) = \sinh(y) \Rightarrow x = y$

$$\frac{e^x - e^{-x}}{2} = \frac{e^y - e^{-y}}{2}$$

$e^x - e^{-x} = e^y - e^{-y}$

$e^x - \frac{1}{e^x} = e^y - e^{-y} \quad / e^x$

$e^{2x} - 1 = e^x(e^y - e^{-y})$

$u = e^x$

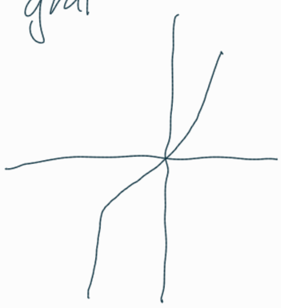
$u^2 - (e^y - e^{-y})u - 1 = 0$

$D = (e^y - e^{-y})^2 + 4 = e^{2y} - 2 + e^{-2y} + 4.$

$= e^{2y} + e^{-2y} + 2 =$

$= (e^y + e^{-y})^2$

graf:



$$u_{1,2} = \frac{-(e^y - e^{-y}) \pm (e^y + e^{-y})}{2}$$

$u_1 = \frac{2e^y}{2} = e^y$

$u_2 = -e^{-y}$

$e^x = e^y$ ali ~~$e^x = -e^{-y}$~~ $\quad >0 \quad <0$

$\Rightarrow x = y$, ker a^x je inj. $\square$ sinh je inj.

inverz.

$\frac{e^x - e^{-x}}{2} = y$

$$e^x - e^{-x} = 2y$$

$$e^{2x} - 1 - (2y)e^x = 0 \qquad u = e^x$$

$$D = 4y^2 + 4$$

$$u_{1,2} = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$$

$$u = y + \sqrt{y^2+1} = e^x$$

$$x = \ln u = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad x \in \mathbb{R}.$$

$$\Longrightarrow Z_{\sinh} = \mathbb{R}.$$

c.)

$$\cosh(x+y) = \frac{e^{x+y} - e^{-y-x}}{2} = \frac{e^x e^y - e^{-y} e^{-x}}{2}$$

$$\cosh(x) \cdot \sinh(y) = \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}}{4}$$

$$\cosh(y) \cdot \sinh(x) = \frac{e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y}}{4}$$

torej $\cosh(x)\sin(y) + \cosh(y)\sinh(x) = \sinh(x+y)$ ...

[domača naloga $\sinh(x \pm y)$]

IV LIMITE FUNKCIJ

$a \in \mathbb{R}$ $\lim_{x \to a} f(x) = L$, če $\forall \varepsilon > 0 \ \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to \infty} f(x) = L$, če $\forall \varepsilon > 0 \ \exists x_0 \in \mathbb{R} \ \ni : x \geq x_0 \Rightarrow |f(x) - L| < \varepsilon$

PRAVILA ZA RAČUNANJE Z LIMITAMI:

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

če $\exists$ ta

$\alpha \in \mathbb{R}$ $\quad \lim_{x \to a} (\alpha f(x)) = \alpha \lim_{x \to a} f(x)$

$$\lim_{x\to a}\big(f(x)g(x)\big)=\lim_{x\to a}f(x)\,\lim_{x\to a}g(x) \qquad (*)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)} \qquad \text{če } \lim_{x\to a}g(x)\neq 0$$

če $\lim_{x\to a}g(x)$ $\exists$ in je f zvezna na okolici $\lim_{x\to a}g(x)$,

velja $\lim_{x\to a}f(g(x))=f\big(\lim_{x\to a}g(x)\big)$ $(*)$

ZNANE limite:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

$$\lim_{x\to 0}(1+x)^{1/x}=e$$

N: let f in g taki fji, da je $\lim_{x\to a}f(x)=\alpha>0$ in

$\lim_{x\to a}g(x)=\beta$, $\alpha,\beta\in\mathbb{R}$.

Dokaži, da velja $\lim_{x\to a}f(x)^{g(x)}=\alpha^{\beta}$. $\lim_{x\to a}e^{\ln\left(f(x)^{g(x)}\right)}=$

$=\lim_{x\to a}e^{g(x)\ln f(x)}$ $\quad \ln f(x)\to \ln\alpha$ $(*)$ $\quad =\lim_{x\to a}e^{\underbrace{g(x)\ln\alpha}_{(*)}}=\lim_{x\to a}e^{\beta\ln\alpha}=$

$=\alpha^{\beta}$

N:

a) $\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1$

b) $\lim_{x\to 0}\frac{1-\cos x}{4x^2}\,\frac{/\cdot(1+\cos x)}{/\cdot(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{4x^2(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{4x^2(1+\cos x)}=$

$=\frac{1}{4}\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{2}=\frac{1}{4}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{2}=\frac{1}{8}$

c) $\lim_{x\to 0}\frac{5\arcsin x}{7x}=\frac{5}{7}\lim_{t\to 0}\frac{t}{\sin t}=\frac{5}{7}\lim_{t\to 0}\frac{1}{\left(\frac{\sin t}{t}\right)}=\frac{5}{7}$

$t=\arcsin x$

$x=\sin t$

d) $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$

sendvič + ocenami:

$0\le\left|x\sin\frac{1}{x}\right|=|x|\cdot\left|\sin\frac{1}{x}\right|\le|x|$

e.)

$$\lim_{x\to 0} \frac{\sin x}{|x|} \qquad -\lim_{x\to -0} \frac{\sin x}{x} = -1$$

$$\lim_{x\to +0} \frac{\sin x}{x} = 1$$

f.) $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2}} \begin{matrix} /:x \\ /:x \end{matrix} = \lim_{x\to\infty} \frac{1}{\sqrt{1+2/x}} = 1$$

g.) $n \in \mathbb{N}$. $$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \left(\frac{x}{e^{x/n}}\right)^n = \lim_{t\to\infty} \left(\frac{nt}{e^t}\right)^n = n^n \left(\lim_{t\to\infty} \frac{t}{e^t}\right)^n$$

$t = x/n$

$e^t = u$: $$\lim_{u\to\infty} \frac{\ln u}{u} = \lim_{u\to\infty} \ln u^{\frac{1}{u}} = \ln 1 = 0$$

h.) $$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$$

$\ln x \to -\infty$, $x \to +0$

i.) $$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \ln \underbrace{\underbrace{(1+x)^{1/x}}_{e}}_{1} = 1$$

j.) $$\lim_{x\to 0} (1+3x)^{\frac{2}{\sin x}} = \lim_{t\to 0} (1+t)^{\frac{2}{\sin t/3}} = \lim_{t\to 0} \Big(\underbrace{(1+t)^{1/t}}_{e}\Big)^{\frac{2t}{\sin(t/3)}} = e^6$$

$3x = t$

$$\lim_{t\to 0} \frac{2t}{\sin(t/3)} = \lim_{x\to 0} \frac{6x}{\sin x} = 6 \cdot 1 = 6$$

$x = t/3$

ZVEZNOST FUNKCIJ

f je zvezna v a, če $\lim_{x\to a} f(x)$ $\exists$ in $= f(a)$.

določi $k \in \mathbb{R}$, da je fja $f: \mathbb{R} \to \mathbb{R}$ zvezna.

predpis: $f(x) = \begin{cases} \frac{x^2-1}{x+1}; & x \in \mathbb{R} \setminus \{-1\} \\ k; & x = -1 \end{cases}$

{elementarne f-je so zvezne na D_f.} f je zvezna na $\mathbb{R} \setminus \{-1\}$.

$$L = \lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = \lim_{x \to -1} x-1 = -2.$$

$$L = -2.$$