Velja: odvedljivost $\Rightarrow$ zveznost.
obrat v splošnem ne velja.

Primer: $f(x) = |x| = \sqrt{x^2}$ JE ZVEZNA, KER JE KOMPOZITUM ZVEZNIH FUNKCIJ.

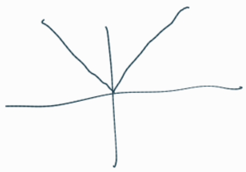
NI ODVEDLJIVA V 0.

$$\frac{f(0+h)-f(0)}{h}$$ } diferenčni kvocient

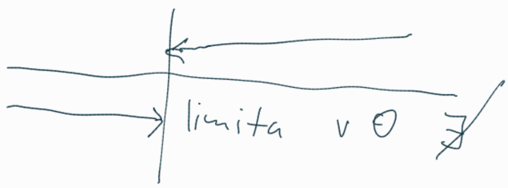
$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \frac{|h|-\cancel{|0|}}{h} = \frac{|h|-0}{h} = \frac{|h|}{h} =$$

$$= \text{signum}(h) = \begin{cases} 1, & h>0 \\ -1, & h<0 \end{cases}$$

graf $|x|$:



signum $h$:




zveznost: „nepretrganost grafa"
odvedljivost: „gladkost grafa"

IZREK: Naj $f, g$ odvedljivi v $x \in \mathbb{R}$.
$\Rightarrow$ tedaj so f-je $f-g$ $f+g$ $f \cdot g$ $f/g$ odvedljive in velja $(f \pm g)' = f' \pm g'$

$$(fg)' = f'g + fg'$$

$$(f/g)' = \frac{f'g - fg'}{g^2}$$

DOKAZ: $(f+g)(x) = f(x) + g(x)$

$$\frac{(f+g)(x+h)-(f+g)(x)}{h} = \frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} =$$

$$= \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

predpostavka:
$f, g$ odv. v $x$ $\leftarrow$ konv. k $f'(x)$

konv. k $g'(x)$

$$(f+g)'(x) = \lim_{h\to x} \frac{(f+g)(x+h)-(f+g)(x)}{h} = f'(x)+g'(x)$$

produkt: predpostavka: $f$ in $g$ sta odvedljivi v $x$.

$$\frac{(fg)(x+h)-(fg)(x)}{h} = \frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$$

$$= \underbrace{\frac{f(x+h)-f(x)}{h}}_{\xrightarrow{h\to 0} f'(x)} \cdot \underbrace{g(x+h)}_{\xrightarrow{h\to 0} g(x)} + \underbrace{\frac{g(x+h)-g(x)}{h}}_{\xrightarrow{h\to 0} g'(x)} \cdot \underbrace{f(x)}_{\xrightarrow{h\to 0} f(x)}$$

$\Downarrow$ $g$ je zvezna, ker je odvedljiva.

$f(x)$ $\Downarrow$ $f$ je odvedljiva $\Rightarrow$ $f$ je zvezna $\Downarrow$ $\lim_{h\to 0} f(x+h) = f(x)$

$$= f'(x)g(x)+g'(x)f(x) = (fg)'(x).$$

$$(f/g)'(x) = \lim_{h\to 0} \frac{(f/g)(x+h)-(f/g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)/g(x+h)-f(x)/g(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x)-f(x)g(x+h)}{h\,g(x+h)g(x)} =$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \frac{g(x)}{g(x+h)g(x)} - \frac{g(x+h)-g(x)}{h} \cdot \frac{f(x)}{g(x+h)g(x)} =$$

$$= \lim \frac{1}{g(x+h)g(x)} \left( \frac{f(x+h)-f(x)}{h} \cdot g(x) - \frac{g(x+h)-g(x)}{h} \cdot f(x) \right) =$$

$$= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} \quad \blacksquare$$

$$\tan' = \frac{\sin' \cos - \sin \cos'}{\cos^2} = \frac{\cos\cos - \sin\cdot(-\sin)}{\cos^2} =$$

$$= \frac{\cos^2 - \sin^2}{\cos^2} = \boxed{\frac{1}{\cos^2} = \tan'}.$$

$$\tan'(x) = \cos^{-2}(x)$$

$(g\circ f)'$ odvod kompozituma:

Izrek: let $f$ odvedljiva v točki $x$
$g$ odvedljiva v točki $f(x)$.

$\Rightarrow$ tedaj je $g\circ f$ odvedljiva v $x$ in velja:

$$(g\circ f)'(x) = g'(f(x))\cdot f'(x).$$

$$\boxed{(g\circ f)(x) := g(f(x)).}$$

Dokaz: $(g\circ f)'(x) = g'(f(x))f'(x)$.

$\delta(h) := f(x+h) - f(x)$

naj bo to točka $a$.

$$\frac{(g\circ f)(x+h) - (g\circ f)(x)}{h} = \frac{g(\underbrace{\boxed{f(x+h)}}_{a+\delta}) - g(\underbrace{\boxed{f(x)}}_{a})}{h} =$$

$$= \frac{g(a+\delta) - g(a)}{h} = \frac{g(a+\delta)-g(a)}{\delta}\cdot \boxed{\delta}/h =$$

$$= \boxed{\frac{g(a+\delta)-g(a)}{\delta}} \cdot \boxed{\frac{\boxed{f(x+h)-f(x)}}{h}} \to f'(x)$$

$h \to 0 \Rightarrow \gamma \to 0$; ker $f$ zvezna, ker $f$ odvedljiva.

$g'(a)$

$$= g'(a) \cdot f'(x) = g'(f(x)) \cdot f'(x) = (g \circ f)'(x)$$

$\square$

ODMOR

Primer:

$$\varphi(x) = \sin(x^2) = (g \circ f)(x) \qquad f(x) = x^2 \quad g(x) = \sin x$$

$$\varphi'(x) = g'(f(x))\, f'(x) = \sin'(x^2) \cdot (x^2)' =$$

$$= \cos x^2 \cdot 2x$$

Primer:

$$\varphi(x) = \sin^2 x = (\sin x)^2 = f(g(x)) \quad \text{kjer sta } f, g \text{ kot prej}$$

$$\varphi'(x) = 2(\sin x) \cos x = 2 \sin x \cos x = \sin 2x$$

↳ „sinus dvojnega kota"

Primer kompozitum treh funkcij

$$\gamma(x) = \sin(e^{x^2}) = \sin\left(e^{(x^2)}\right) = (g \circ h \circ f)(x)$$

$h(x) = e^x = \exp(x)$

$g(x) = \sin x$

$f(x) = x^2$

$$\gamma'(x) = \cos(e^{x^2})\, e^{x^2}\, 2x$$

$$\exp'(x) = \exp(x)$$

Fja $f: I \subseteq \mathbb{R} \to \mathbb{R}$ je zvezno odvedljiva na $I$, če $f'$ na $I$ obstaja in je tam zvezen.

(tdb $g: x \mapsto f'(x)$ zvezna)

Primer: fja, ki je v 0 odvedljiva, ni pa zvezno odvedljiva:

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ;\ x\neq 0\\ 0 & ;\ x=0\end{cases}$$

dokažimo:

$$f'(x)=\begin{cases} 2x\sin 1/x - \cos 1/x & ;\ x\neq 0 \\ 0 & ;\ x=0\end{cases}$$

$$f'(0)=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{h^2\sin 1/h - 0}{h} = h\underbrace{\sin\frac{1}{h}}_{\text{omejena z 1!}} = 0, \quad h\to 0$$

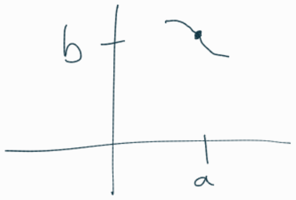
Dokaz: $f'$ je zvezna v $0 \Leftrightarrow \lim_{x\to 0} f'(x)=f'(0)$

toda $f'(x)=\underbrace{2x\sin 1/x}_{\to\, 0} - \underbrace{\cos\frac{1}{x}}_{\text{nima limite proti } 0}$

$\cos 1/x$

IZREK: odvod inverza: let $f$ strogo monotona v okolici $a$, v $a$ odvedljiva in naj bo $f'(a)\neq 0$

tedaj bo inverzna funkcija, definirana v okolici $b=f(a)$ v $b$ odvedljiva in $(f^{-1})'(b)=\dfrac{1}{f'(a)}$




strogo monotona: $\exists f^{-1}$!

$$f(x)=s \Leftrightarrow x=f^{-1}(s)$$

$$\Rightarrow f^{-1}(f(x))=x$$

Formula za odvod kompozituma:

$$(f^{-1})'(f(x))\cdot f'(x)=(x)'$$

$$(f^{-1})'(f(x))\,f'(x)=1 \Rightarrow (f^{-1})'(f(x))=\frac{1}{f'(x)}$$

za $\forall x$ blizu $a$

v posebnem: $(f^{-1})'(b)=\dfrac{1}{f'(a)}$ $(*)$

Primeri:

1.) $g(x)=\sqrt[n]{x}$ $n\in\mathbb{N}$, $x>0$

velja $g=f^{-1}$ za $f(x)=x^n$

$(*)$

$$(f^{-1})'(f(x))=\frac{1}{f'(x)}$$

$$g'(x^n)=\frac{1}{n\cdot x^{n-1}}$$

$$t:=x^n \Rightarrow x=t^{1/n}$$

$$g'(t)=\frac{1}{n(t^{1/n})^{n-1}}=\frac{1}{nt^{1-1/n}}=\frac{1}{n}t^{1/n-1}$$

$\Rightarrow$ za $h(x)=\sqrt[n]{x^m}$ je

$$h'(x)=\frac{m}{n}x^{\frac{m}{n}-1}$$

$$\Rightarrow \boxed{(x^\alpha)'=\alpha x^{\alpha-1} \quad \text{za } \forall x>0,\ \forall \alpha\in\mathbb{R}.}$$

ODMOR (cont. Primeri)

2.) logaritem, inverz eksponentne funkcije.

$$g'(f(x))=1/f'(x)$$

$$g'(e^x)=1/e^x \qquad e^x=:t$$

$$g'(t)=\frac{1}{t}$$

$$\log'(t)=1/t$$

3. $g(x)=\arcsin(x)$; $x\in[-1,1]$

velja $g=f^{-1}$, kjer $f(x)=\sin x$ $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$$g'(f(x))=\frac{1}{f'(x)} \qquad g'(\sin x)=\frac{1}{\cos x}$$

$t:=\sin x$

vemo $\sin^2x+\cos^2x=1$,
zato $\cos^2x=1-\sin^2x$
ker $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$, vemo
$\cos\geq 0$,
torej $\cos x=$
$\sqrt{1-\sin^2x}$.

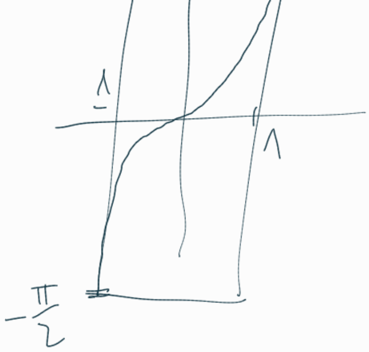
$$g'(t)=\frac{1}{\sqrt{1-\sin^2x}}=\frac{1}{\sqrt{1-t^2}}$$

$t\in[-1,1]$

Skica:

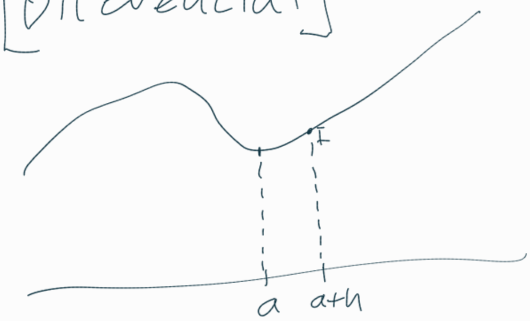
arcsin:

$\arcsin'(1)=\infty$




$\arcsin'(-1) = \infty$

[Diferencial]




iščemo najboljši linearni

od h odv.

približek za $f(a+h) - f(a)$

Torej funkcijo oblike

$\varphi(h) = \text{const}\cdot h$

če je $f$ odvedljiv, je $\text{const} = f'(a)$.

Označimo $\varphi = \underbrace{df(a)}$ ime linearne funkcije, konstruirane iz $f$ in $a$.

funkcija $df(a)$ se imenuje diferencial fje $f$ v točki $a$. Velja:

$$\lim_{h\to 0} \frac{\boxed{f(a+h) - f(a)} - (df(a))(h)}{h} = 0 = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} - \frac{df(a)\cancel{h}}{\cancel{h}} =$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} - df(a) =$$

Dokaz sledi iz $(df(x))(h) = f'(a)h$ in definicije $f'(a)$.

$$= f'(h) - df(a) = 0$$

$$\Downarrow$$

$$f'(h) = df(a)$$

Torej $f(a+h) - f(a) \doteq \underbrace{df(a)(h)}_{\text{najboljši linearni približek za}}$

$f(a+h) - f(a)$.