Rešitev osme domače naloge Linearne Algebre

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Povzetek

Za boljšo preglednost sem svoje rešitve domače naloge prepisal na računalnik. Dokumentu sledi še rokopis. Naloge je izdelala asistentka Ajda Lemut.

1. Dokaži, da je $[(x,y,z),(u,v,w)] = 2xu - yu - xv + 2yv - zv - yw + zw$ skalarni produkt in ugotovi, ali je

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

normalna preslikava glede na $[\cdot,\cdot]$.

Rešitev Predpostavljam polje $\mathbb{R}$ in vektorski prostor $V = \mathbb{R}^3$, saj v kompleksnem to ni skalarni produkt (protiprimer pozitivne definitnosti je $[(1,1,1+i),(1,1,1+i)] = 2$). $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ je skalarni produkt, če zadošča naslednjim lastnostim. Dokažimo jih za $[\cdot,\cdot]$.

(a) $\forall v \in V : v \neq 0 \Rightarrow \langle v, v\rangle > 0$

$$[(x,y,z),(x,y,z)] = 2x^2 - 2xy + 2y^2 - 2yz + z^2 = 2\left(x^2 - xy + y^2\right) - 2yz + z^2 =$$

$$= 2\left(\left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4}\right) - 2yz + z^2 = 2\left(x - \frac{y}{2}\right)^2 - \frac{3y^2}{4} - 2yz + z^2 =$$

$$= 2\left(x - \frac{y}{2}\right)^2 + \left(\frac{\sqrt{3}y}{\sqrt{2}} - \frac{z\sqrt{2}}{\sqrt{3}}\right)^2 + \frac{z^2}{3} \geq 0$$

Sedaj poiščimo ničle. Fiksirajmo poljubna y, z in uporabimo obrazec za ničle kvadratne enačbe:

$$x_{1,2} = \frac{2y \pm \sqrt{4y^2 - 8\left(2y^2 - 2yz + z^2\right)}}{4}$$

Iščemo pozitivne diskriminante.

$$4y^2 - 8\left(2y^2 - 2yz + z^2\right) = -12y^2 + 16yz - 8z^2 = 4$$

Fiksirajmo poljuben z. Vodilni koeficient kvadratne enačbe je negativen. Uporabimo obrazec:

$$y_{1,2} = \frac{-16z \pm \sqrt{256z^2 - 384z^2 = -128z^2}}{-24}$$

Diskriminanta je nenegativna $\Leftrightarrow z = 0$. Torej $z = 0$, zato $y = 0$ in tudi $x = 0$ glede na obrazce. Skalarni produkt je res pozitivno definiten.

(b) $\forall v, u \in V : \langle v, u\rangle = \langle u, v\rangle$

$$[(u,v,w),(x,y,z)] = [(x,y,z),(u,v,w)]$$

$$2ux - vx - uy - 2vy - wy - vz + wz = 2xu - yu - xv + 2yv - zv - vz + wz$$

$$0 = 0$$

Skalarni produkt je res simetričen.

(c) $\forall \alpha_1, \alpha_2 \in \mathbb{C} \forall u_1, u_2, v \in V : \langle \alpha_1 v_1 + \alpha_2 v_2, v\rangle = \alpha_1 \langle u_1, v\rangle + \alpha_2 \langle u_2, v\rangle$

$$[\alpha\left(\left(x_1, y_1, z_1\right)+\left(x_2, y_2, z_2\right)\right),(u, v, w)]=$$

$$=2\alpha\left(x_1+x_2\right)u-\alpha\left(y_1+y_2\right)u-\alpha\left(x_1+x_2\right)v+2\alpha\left(y_1+y_2\right)v-\alpha\left(z_1+z_2\right)v-\alpha\left(y_1+y_2\right)w+\alpha\left(z_1+z_2\right)w=$$
$$=\alpha\left(2\left(x_1+x_2\right)u-\left(y_1+y_2\right)u-\left(x_1+x_2\right)v+2\left(y_1+y_2\right)v-\left(z_1+z_2\right)v-\left(y_1+y_2\right)w+\left(z_1+z_2\right)w\right)=$$
$$=\alpha\left(2x_1u+2x_2u-y_1u-y_2u-x_1v-x_2v+2y_1v+2y_2v-z_1v-z_2v-y_1w-y_2w+z_1w+z_2w\right)=$$
$$=\alpha\left(2x_1u-y_1u-x_1v+2y_1v-z_1v-y_1w+z_1w\right)+\alpha\left(2x_2u-y_2u-x_2v+2y_2v-z_2v-y_2w+z_2w\right)=$$
$$=\alpha\left[\left(x_1, y_1, z_1\right),(u, v, w)\right]+\alpha\left[\left(x_2, y_2, z_2\right),(u, v, w)\right]$$

Skalarni produkt je res homogen in linearen v prvem faktorju.

Po definiciji A normalna $\Leftrightarrow A^*A = AA^*$. Izračunajmo torej matriko A^*.

- Na predavanjih 2024-05-08 smo dokazali, da za vsak skalarni produkt $[u, v]$ obstaja taka pozitivno definitna matrika M, da velja $[u, v] = \langle u, Mv\rangle = u^*v$, kjer je $\langle\cdot,\cdot\rangle$ standardni skalarni produkt.
- Na predavanjih 2024-04-17 smo dokazali, da $[L^*]_{C\leftarrow B} = \left([L]_{B\leftarrow C}\right)^*$, torej $PLP^{-1} = \left(P^{-1}L^*P\right)^*$.
- Izpeljimo predpis za A^* pri podani matriki A in skalarnem produktu $[\cdot,\cdot]$ s pripadajočo matriko M:

$$\left[A^*x, y\right]=[x, Ay]\text{, uporabimo prvo točko:}$$

$$\left\langle A^*x, My\right\rangle=\langle x, MAy\rangle\text{, pišimo } z=My\text{:}$$
$$\left\langle A^*x, z\right\rangle=\left\langle x, MAM^{-1}z\right\rangle\text{, upoštevajmo drugo točko:}$$
$$\left\langle A^*x, z\right\rangle=\left\langle M^{-1}A^{\square}Mx, z\right\rangle\text{, kjer je } A^{\square} \text{ adjungacija } A \text{ pri standardnem skalarnem produktu}$$

$$\Rightarrow A^*=M^{-1}A^{\square}M=M^{-1}\overline{A}^TM \overset{A\in M(\mathbb{R})}{=} M^{-1}A^TM$$

- Potrebujemo še matriko skalarnega produkta.

$$\langle(x, y, z), M(u, v, w)\rangle=\left[\begin{array}{lll}x & y & z\end{array}\right]\left[\begin{array}{lll}m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33}\end{array}\right]\left[\begin{array}{l}u \\ v \\ w\end{array}\right]=\left[\begin{array}{lll}x & y & z\end{array}\right]\left[\begin{array}{l}um_{11}+vm_{12}+wm_{13} \\ um_{21}+vm_{22}+wm_{23} \\ um_{31}+vm_{32}+wm_{33}\end{array}\right]=$$

$$=\begin{array}{ccccc} & xum_{11} & xvm_{12} & xwm_{13} & + \\ + & yum_{21} & yvm_{22} & ywm_{23} & + \\ + & zum_{31} & zvm_{32} & zwm_{33} & \end{array}=[(x, y, z),(u, v, w)]=2xu-yu-xv+2yv-zv-yw+zw\text{, torej}$$

$$M=\left[\begin{array}{ccc}2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1\end{array}\right]\text{, njen inverz pa je } M^{-1}=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3\end{array}\right]$$

- Izračunamo A^* po formuli $A^* = M^{-1}A^TM$ in preverimo $A^*A = AA^*$:

$$A^*=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3\end{array}\right]\left[\begin{array}{ccc}0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & -1\end{array}\right]\left[\begin{array}{ccc}2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1\end{array}\right]=\left[\begin{array}{ccc}-1 & 2 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & -2\end{array}\right]$$

$$A^*A=\left[\begin{array}{ccc}1 & -2 & 3 \\ 1 & -3 & 5 \\ 2 & -10 & 14\end{array}\right] \neq\left[\begin{array}{ccc}8 & -6 & 2 \\ -2 & 3 & -1 \\ 3 & -2 & 1\end{array}\right]=AA^*$$

A ni normala matrika.

- Da preverimo pravilnost matrike A^*, lahko napravimo preizkus:

```
sage: A = Matrix([[0, 2, -2], [0, 1, 0], [-1, 2, -1]])
sage: AA = Matrix([[-1, 2, -1], [-2, 3, -1], [-6, 6, -2]])
sage: def sp(a,b): return 2*a[0]*b[0]-a[1]*b[0]-a[0]*b[1]+2*a[1]*b[1]-a[2]*b[1]-a[1]*b[2]+a[2]*b[2]
sage: xyz = vector([var("x"), var("y"), var("z")])
sage: uvw = vector([var("u"), var("v"), var("w")])
sage: (sp(A*xyz, uvw) == sp(xyz, AA*uvw)).expand()
v*x - w*x + 3*u*y - 2*v*y + w*y - 4*u*z + 3*v*z - w*z == v*x - w*x + 3*u*y - 2*v*y + w*y - 4*u*z + 3*v*z - w*z
sage: bool((sp(A*xyz, uvw) == sp(xyz, AA*uvw)).expand())
True
sage:
```

Slika 1: Preizkus s programom SageMath.

2. Pokaži $A : V \to V$ je normalna $\Leftrightarrow AA^* - A^*A$ je pozitivno semidefinitna.

Rešitev

- Definiciji:
  - $A : V \to V$ je normalna $\Leftrightarrow A^*A = A^*$
  - $A : V \to V$ je pozitivno semidefinitna $\Leftrightarrow A = A^* \wedge \forall v \in V : \langle Av, v\rangle \geq 0$
- $(\Rightarrow)$ Po predpostavki velja $AA^* = A^*A \Rightarrow AA^* - A^*A = 0$.

$$AA^* - A^*A \stackrel{?}{=} (AA^* - A^*A)^* \Leftrightarrow 0 = 0^*$$

$$\langle (AA^* - A^*A)\, v, v\rangle = \langle 0v, v\rangle \stackrel{\text{homogenost}}{=} 0\, \langle v, v\rangle = 0 \geq 0$$

- $(\Leftarrow)$Po predpostavki velja $(AA^* - A^*A)^* = AA^* - A^*A$

3. Naj bo $w_1 = (1, 1, 1, 1)$, $w_2 = (3, 3, -1, -1)$ in $y = (6, 0, 2, 0)$.
   (a) Poišči ortonormirano bazo za $W = \mathrm{Lin}\,\{w_1, w_2\}$ glede na standardni skalarni produkt.
   (b) Izrazi y kot vsoto vektorja iz W in vektorja iz $W^\perp$.

Rešitev

(a) Uporabimo Gram-Schmidtov postopek in sproti normiramo bazne vektorje:

$$v_1 = (1, 1, 1, 1)\,/2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\tilde{v_2} = (3, 3, -1, -1) - \langle (3, 3, -1, -1)\,, v_1\rangle\, v_1 = (3, 3, -1, -1) - \left\langle (3, 3, -1, -1)\,, \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\rangle \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) =$$

$$= (3, 3, -1, -1) - 2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = (2, 2, -2, -2)\,, \qquad v_2 = \tilde{v_2}/\,||\tilde{v_2}|| = \tilde{v_2}/4 = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$$

Baza za W je $B = \{v_1, v_2\} = \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)\right\}$.

(b) Uporabimo fourierov razvoj y po B.

$$y = \sum_{i=1}^{n} \frac{\langle y, v_i\rangle}{\langle v_i, v_i\rangle} v_i = \left\langle (-6, 0, 2, 0)\,, \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\rangle v_i + \left\langle (-6, 0, 2, 0)\,, \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)\right\rangle \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}\right) =$$

$$= 4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + 2\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}\right) = (2, 2, 2, 2) + (1, 1, -1, -1)$$

4. Poišči singularni razcep matrike

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Rešitev

- Iščemo U, Σ in V, da velja $A = U\Sigma V^*$.
- Diagonalci Σ so singularne vrednosti A. Singularne vrednosti A so koreni lastnih vrednosti
- A^*A, torej $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = 0$.

$$A^*A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Stolpci V so ortonormirana baza jedra $A^*A - \sigma^2 I$ za vse singularne vrednosti σ.

$$A^*A - 4I = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = 0 \Rightarrow v_1 = (0, 1, 0)$$

$$A^*A - 1I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow v_2 = (1, 0, 0)$$

$$A^*A - 0I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = y = 0 \Rightarrow v_3 = (0, 0, 1)$$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Stolpci U so ortonormirana baza in velja $\forall i \in \{1..\operatorname{rang} A\} : u_i = \sigma_i^{-1} A v_i$. Stolpične vektorje $v_{\operatorname{rang} A+1}, \ldots, v_m$ najdemo tako, da dopolnimo $v_1, \ldots, v_{\operatorname{rang} A}$ do ONB.

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Dobljene matrike zmnožimo, s čimer potrdimo veljavnost singularnega razcepa:

$$U\Sigma V^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$$

Rokopisi, ki sledijo, naj služijo le kot dokaz samostojnega reševanja. Zavedam se namreč njihovega neličnega izgleda.